Product Topology

DR. SHEO KUMAR SINGH
Assistant Professor,
Department of Mathematics
School of Physical Sciences,
Mahatma Gandhi Central University, Motihari, Bihar-845401.
Email: sheokumarsingh@mgcub.ac.in

Product Topology

Let X and Y be topological spaces. Then

- the collection $\mathfrak{B} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ is a base for a topology on $\mathrm{X} \times Y$.

- the topology generated by the base $\mathfrak{B}$ is said to be the product topology on $\mathrm{X} \times Y$.

- this topology is also said to be the box topology on $\mathrm{X} \times Y$.

Another way of describing product topology

Let X and Y be topological spaces. Then

- $p_1\colon \mathrm{X} \times Y \to X$ and $p_2\colon \mathrm{X} \times Y \to Y$ defined respectively, by $p_1(x, y) = x$ and $p_2(x, y) = y$ are the projection maps in the components X and Y.
- the collection $\mathcal{S} = \{{p_1}^{-1}(U) |\ U \text{ is open in } X\} \cup \{{p_2}^{-1}(V) |\ V \text{ is open in } Y\}$ is a subbase for a topology on $\mathrm{X} \times Y$.
- the topology generated by the subbase $\mathcal{S}$ is said to be the product topology on $\mathrm{X} \times Y$.
- the basis element $U \times V = {p_1}^{-1}(U) \cap {p_2}^{-1}(V)$.

Some important results

Proposition 1:

Product of two Hausdorff spaces is Hausdorff.

Proof: Let X and Y be two Hausdorff spaces. Let (x_1, y_1) and (x_2, y_2) be two arbitrary *distinct* elements of $\mathrm{X} \times Y$. So, $x_1 \neq x_2$ or $y_1 \neq y_2$. Without any loss of generality, let $x_1 \neq x_2$, then as X is Hausdorff, so there must be two *disjoint* open sets U and V of X such that $x_1 \in U$ and $x_2 \in V$. But then $\mathrm{U} \times Y$ and $\mathrm{X} \times V$ are also disjoint. Clearly, $\mathrm{U} \times Y$ and $\mathrm{X} \times V$ are (basic) open sets of $\mathrm{X} \times Y$ with $(x_1, y_1) \in \mathrm{U} \times Y$ and $(x_2, y_2) \in \mathrm{X} \times V$. Thus, $\mathrm{X} \times Y$ is also Hausdorff. ■

Thus, hausdorffness is a productive property (a property P is said to be productive if each component has the property P then the product space also has the same property P).

Proposition 2:

A topological space X is Hausdorff if and only if the diagonal $\Delta_X = \{(x,x) | x \in X\}$ is closed in $X \times X$.

Proof: Let X be a Hausdorff space. We will show that complement of Δ_X is open in $\mathrm{X} \times X$. So, let $(x,y) \in \mathrm{X} \times X$-$\Delta_X$. Then, $(x,y) \notin \Delta_X$. Hence, $x \neq y$. But then, being X Hausdorff, there must be *disjoint* open sets U and V of X such that $x \in U$ and $y \in V$. Clearly, $\mathrm{U} \times V$ is open in $\mathrm{X} \times X$ and $(\mathrm{U} \times V) \cap \Delta_X = \emptyset$. So, $\mathrm{U} \times V \subseteq X \times X$-$\Delta_X$. Thus, $(x,y) \in \mathrm{U} \times V \subseteq \mathrm{X} \times X - \Delta_X$, which implies that $X \times X$-Δ_X is a neighborhood of (x,y). Hence, $X \times X$-Δ_X is open and so, Δ_X is closed in $X \times X$.

Conversely, let Δ_X be closed in $X \times X$. To show X Hausdorff, let $x, y \in X$ with $x \neq y$. Hence, $(x,y) \notin \Delta_X = \overline{\Delta_X}$. But then, there must be some basic open set $\mathrm{U} \times V$ in $X \times X$ with $(x,y) \in \mathrm{U} \times V$ such that $(\mathrm{U} \times V) \cap \Delta_X = \emptyset$, whereby it follows that $\mathrm{U} \cap V = \emptyset$. Thus, X is Hausdorff. ■

References:

- James R. Munkres, *Topology*, 2nd ed., PHI.
- Colin Adams and Robert Franzosa, *Introduction to Topology: Pure and Applied*, Pearson.

THANK YOU