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Title: Electromagnetic Theory

Topics: Propagation of Electromagnetic Waves in Ionized gas and Polarization of Electromagnetic Waves

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Topic - Propagation of Electromagnetic wave in Ionised gas

A simple Model for Dynamical Conductivity

If the field frequency developed across a conductor is raised, the conducting electrons due to their inertia, follow the field with raising difficulty. It suggests a decrease in the conductivity with increasing frequency. To understand this, let us suppose the simple model given by Drude. According to this model a metal contains a certain number (say) n_0 of electrons per unit volume free to move under the action of applied electric fields; but subject to damping force due to collision.

The collisions occur between electrons and Lattice vibrations, Lattice imperfections, and impurities.

If b is the damping constant, then the damping force may be written as -

$$F_{damp} = -\, bmv \quad \text{--------} \quad (i)$$

If E is electric field applied across a conductor, then from Newton's 2nd law, the eq. of motion of conducting electron-

$$m\frac{dv}{dt} = eE - bmv \quad \text{------} \quad (ii)$$

For rapidly oscillating fields, the displacement of the electron is small compared to a wavelength, which is $E = E_0 e^{-i\omega t}$

Now from eq. (ii) $\quad m\frac{dv}{dt} = eE_0 e^{-i\omega t} - bmv \quad - \quad (iii)$

Here E_0 is the electric field at the average position of the electron

Now, the current density J -

$$J = n_0 e v \quad \text{or} \quad v = \frac{J}{n_0 e}$$

from eq. (iii)

$$m \frac{d}{dt}\left(\frac{J}{n_o e}\right) = eE_o e^{-i\omega t} - bm\left(\frac{J}{n_o e}\right) \quad \text{-----} (iv)$$

Simplifying and rearranging this eq.

$$m\frac{dJ}{dt} + bmJ = n_o e^2 E e^{-i\omega t}$$

Also for time varying current density $J = J_o e^{-i\omega t}$

Now from eq. (iv) $m\frac{d}{dt}\left(J_o e^{-i\omega t}\right) + bmJ_o e^{-i\omega t} = n_o e^2 E_o e^{-i\omega t}$

$$m J_o(-i\omega) e^{-i\omega t} + bmJ_o e^{-i\omega t} = n_o e^2 E_o e^{-i\omega t}$$

$$-im\omega J + bmJ = n_o e^2 E$$

$$\Downarrow$$

$$J = \frac{n_o e^2 E}{m(b - i\omega)} \quad \text{-----} (v)$$

If we compare this with $J = \sigma E$ – we find that –

$$\boxed{\sigma = \frac{n_o e^2}{m(b - i\omega)}} \quad \text{-----} (vi)$$

In a metal such as a copper where $n_o \simeq 1\times10^{28}$ electrons/m^3, $\sigma = 5\times10^7$ S/m has an empirical damping constants $b \simeq 3\times10^{13}\ sec^{-1}$.

→ It is clear that for frequencies of the order of or smaller than, microwave frequencies ($\sim10^{10}\ sec^{-1}$) the electrical conductivity is essentially real (current is in phase with the applied field) and it is independent of frequency ($\omega \simeq 0$) – and thus it takes the form $\sigma = \frac{n_o e^2}{mb}$ ..... (vii)

This is well known Lorentz - Drude expression for conductivity

* At higher frequencies, however, the conductivity is complex and depends markedly on frequency in a manner defined in equatn (vi).

* Maxwell's equations and eq. of EM waves in ionised me

In some of cases of ionised gases when the pressure is quit low such as the ionosphere or a plasma, we may suppose that there a no collisions and hence no energy losses (damping constant $b=0$) so that the conductivity σ defined from eq. (VI) becomes purely imaginary and it is thus given by $\sigma = -\frac{n_0 e^2}{mi\omega} \simeq \frac{i n_0 e^2}{m\omega}$.

Now the differential form of Maxwell's eqs.

$$\left.\begin{aligned} \nabla\cdot D &= 0 \\ \nabla\cdot B &= 0 \\ \nabla\times E &= -\frac{\partial B}{\partial t} \\ \nabla\times H &= J + \frac{\partial D}{\partial t} \end{aligned}\right\} \quad \begin{aligned} &\text{with } J = \sigma E \quad \text{and } \sigma = \frac{i n_0 e^2}{m\omega} \\ &B = \mu H \\ &D = \varepsilon E \qquad \varepsilon = \varepsilon_0 \\ &\rho_v = 0 \qquad \mu = \mu_0 \end{aligned}$$

$$\nabla\times E = -\mu_0 \frac{\partial H}{\partial t} \quad -(a) \qquad \nabla\cdot E = 0 \quad -(c)$$

$$\nabla\times H = \sigma E + \varepsilon_0 \frac{\partial E}{\partial t} \quad -(b) \qquad \nabla\cdot H = 0 \quad -(d)$$

→ Now taking curl of eq. (a)

$$\nabla\times\nabla\times E = -\mu_0 \frac{\partial}{\partial t}(\nabla\times H)$$

$$(\nabla\cdot E)\nabla - (\nabla^2 E) = -\mu_0 \frac{\partial}{\partial t}\left(\sigma E + \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\nabla^2 E - \mu_0 \sigma \frac{\partial E}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} = 0 \quad -(\text{VIII})$$

Similarly taking curl of eq. (b).

and we get $$\nabla^2 H - \mu_0 \sigma \frac{\partial H}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 H}{\partial t^2} = 0 \quad -(\text{VIII})$$

Eqs. (VIII) and (IX) represent wave eqs. in terms of electro-magnetic field vectors E and H in ionised medium.

These equations are vector equations of similar form, therefore each of six components of E and H separately satisfies the same scalar wave equation of the form-

$$\nabla^2 f - \mu_0 \sigma \frac{\partial f}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 f}{\partial t^2} = 0 \quad - \quad (X)$$

where f is a scalar and can stand for any of six components of E and H.

→ The plane wave solutions of eqs those are mentioned above -

$$\left.\begin{aligned} E &= E_0 e^{ik\cdot r - i\omega t} \\ H &= H_0 e^{ik\cdot r - i\omega t} \\ f &= f_0 e^{ik\cdot r - i\omega t} \end{aligned}\right\} \quad - \quad (XI)$$

where E_0 and H_0, f_0 are complex amplitudes which are constant in space and time, and k is a vector quantity known as a wave vector or wave propagation vector and defined as

$$K = k\cdot\hat{n} = \frac{2\pi}{\lambda}\cdot\hat{n} = \frac{\omega}{v}\cdot\hat{n}$$

here $\hat{n}$ is a unit vector along k and v is phase velocity of the wave. , Now from eqs (XI) $\nabla^2 f = -K^2 f$

and $\frac{\partial f}{\partial t} = -i\omega f$ and $\frac{\partial^2 f}{\partial t^2} = -\omega^2 f$

Now putting these into eq. (X)

$$-K^2 f + i\omega\mu_0\sigma f + \mu_0\varepsilon_0\omega^2 f = 0$$

$$(K^2 - i\omega\mu_0\sigma - \mu_0\varepsilon_0\omega^2) f = 0$$

As f is an arbitorary component of field vector, hence above eq. holds only if $K^2 - i\omega\mu_0\sigma - \mu_0\varepsilon_0\omega^2 = 0$

$$K^2 = \mu_0\varepsilon_0\omega^2\left[1 + \frac{i\sigma}{\omega\varepsilon_0}\right] \quad -$$

Substituting the value of σ in Last equation.

$$K^2 = \mu_0 \varepsilon_0 \omega^2 \left[1 - \frac{n_0 e^2}{m \varepsilon_0 \omega^2}\right] = \frac{\omega^2}{c^2}\left[1 - \frac{n_0 e^2}{m \varepsilon_0 \omega^2}\right]$$

$$K^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_p^2}{\omega^2}\right] \quad \text{where } \omega_p^2 = \frac{n_0 e^2}{m \varepsilon_0}$$

- (XII)

ω_p is known as the plasma frequency. and $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

* If n is the refractive index, then $n = \frac{c}{v}$

$\therefore K = \frac{\omega}{v} = \frac{n\omega}{v}$

$K^2 = \frac{n^2\omega^2}{c^2}$ from equation - (XII).

the refractive index of a plasma medium -

$$\boxed{n^2 = 1 - \frac{\omega_p^2}{\omega^2}}$$

* For high frequency region $\omega > \omega_p$, the refractive index n $\left(n = \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}\right)$ is real and therefore waves propagate free

* For low frequency region $\omega < \omega_p$, the refractive index, n is purely imaginary. As a result such EM waves incident on a plasm will be reflected from the surface.

$\because \quad K = \frac{n\omega}{c}$

if we replace in in place of n in above eq

$K = \frac{in\omega}{c}$

Now, the sols for E and H for low frequency regime may be expressed as $E = E_0 e^{ik(\hat{n}\cdot r) - i\omega t}$

$$E = E_0 e^{-(\omega n/c)(\hat{n}\cdot r)} e^{-i\omega t} \quad \text{- (XIII)}$$

$$\text{and } H = H_0 e^{-\omega n/c\,(\hat{n}\cdot r)} e^{-i\omega t} \quad \text{- (XIV)}$$

These equations represent that within the electromagnetic field vectors E and H will fall off exponentially with distance from the surface.

* The skin depth or penetration depth for the plasma

The skin depth $\delta_{plasma} = \frac{1}{\beta} = \frac{1}{\left(\frac{\omega n}{c}\right)} \frac{1}{\left[\frac{\omega_p^2}{\omega^2} - 1\right]^{1/2}}$

$\because \beta = (\omega n/c)$

$\therefore \delta_{plasma} = \frac{c}{\sqrt{\omega_p^2 - \omega^2}} = \left(\frac{c}{\omega_p}\right) \quad (\because \omega << \omega_p)$

⇒ At the laboratory scale plasma densities are in the range $n_0 \simeq 10^{10} - 10^{12}$ electrons/m^3. This means that plasma frequency $\omega_p \simeq 6\times10^{10} - 6\times10^{12}$ sec^{-1}), so that the typical penetration depths are of the order of 0.5 cm to 5×10^{-3} cm for low frequency fields.

* Critical frequency for propagation of EM waves in plasma-

→ we know that the transmission is possible in plasma only when the refractive index is real.

The refractive index n is given by $n^2 = 1 - \frac{\omega_p^2}{\omega^2}$

or $n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$ where $\omega_p^2 = \frac{n_0 e^2}{m \epsilon_0}$

If refractive index n is real, then $\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \geq 0$

or $\omega_0^2 \geq \omega_p^2$ if ω_0 is the critical angular frequency for propagation of EM waves in plasma then $\quad \omega_0 = \omega_p = \sqrt{\frac{n_0 e^2}{m \varepsilon_0}}$ and thus critical frequency

$$F_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{n_0 e^2}{m \varepsilon_0}} \Rightarrow \boxed{F_0 = \frac{1}{2\pi}\sqrt{\frac{n_0 e^2}{m \varepsilon_0}}}$$

$$F_0 = \frac{1}{2\times 3.14}\sqrt{\frac{n_0 \times (1.6\times 10^{-19})^2}{(9\times 10^{-31} \times 8.85\times 10^{-12}}} = 9\sqrt{n_0}$$

* $\boxed{F_0 = 9\sqrt{n_0}}$

Numerical-1 - calculate the plasma frequency and maximum - peneteration depth for a plasma containing 10^{18} electrons/m^3.

Solution - Plasma frequency $F_0 = 9\sqrt{n_0}$

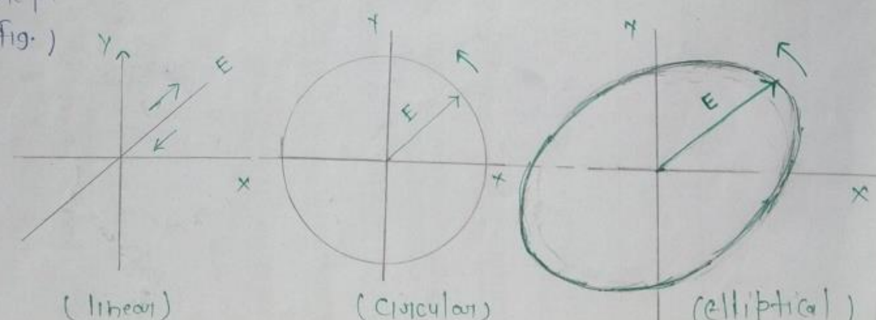
$$= 9\times\sqrt{10^{18}} = 9\times 10^9 \text{ Hz}$$

$$F_0 = 9000 \text{ MHz}$$ Ans.

* **Wave polarization** - It is a common practice to present an EM wave by its polarization. Polarization is an important characteristic of an EM wave, and the concept has been developed to represent the various types of electric field variation and orientation. The polarization of an EM wave depends on the transmitting antenna or source. It is determined by the direction of the electric field for fields consisting more than one component.

⇒ Polarization may be regarded as the locus of the tip of the electric field (in a plane perpendicular to the direction of wave propagation) at a given point as a function of time.

⇒ There are three types of polarization:- Linear or plane, circular, and elliptical. They defined that the tip of the electric field can represent a straight line, a circle, or an ellipse with time (see fig.)



⇒ Wave polarization is important for radio and TV broadcasting. Amplitude modulation (AM) radio broadcasting is with polarization vertical to the earth's surface, while frequency mod. (FM) broadcasting is generally circularly polarized.

⇒ A uniform plane wave is linearly polarized if it has only one component or when its transverse components are in phase.

⇒ Suppose a wave travelling in the +z-direction, we

$E_x = E_{ox} \cos(\omega t - \beta z + \phi_x)$ — (i)

$E_y = E_{oy} \cos(\omega t - \beta z + \phi_y)$ — (ii)

E_{ox} and E_{oy} are real. The superpositioned wave –

$$E = E_{ox} \cos(\omega t - \beta z + \phi_x)\, \hat{a}_x + E_{oy} \cos(\omega t - \beta z + \phi_y)\, \hat{a}_y$$

is linearly polarized when the phase difference $\Delta\phi$ is –

$$\Delta\phi = \phi_y - \phi_x = n\pi, \quad n = 0, 1, 2, 3 ..$$

This allows the two components to maintain the similar ratio at all times. If we detect the wave in the direction of propagation (say z direction), we will find that the tip of the electric fie[ld] follows a line – that is why known as linear polarization.

⇒ Linearly polarized plane waves can be generated by simple anten[nas] (such as dipole antennas) or lasers.

* Circular polarization takes place when the x- and y- components are the similar in magnitude ($E_{ox} = E_{oy} = E_o$) and the phase difference b/w them is an odd multiple of $\pi/2$, that is

$$\Delta\phi = \phi_y - \phi_x = \pm (2n+1)\pi/2, \quad n = 0, 1, 2, 3 ..$$

suppose, $E_x = E_o \cos(\omega t - \beta z)$ — (iii)

$E_y = E_o \cos(\omega t - \beta z + \pi/2)$ — (iv)

* Also another form of circularly polarized wave (phasor) form.

$$E(z,t) = \mathrm{Re}\{ E_o e^{i\omega t} e^{-i\beta z} [a_x + e^{\pm i\pi/2} a_y] \}$$

$e^{\pm i\pi/2} = \pm j$ so $E_s = E_o (a_x \pm i a_y) e^{-i\beta z}$

where the plus sign is used for left circularly polarized wave and minus sign for right circular polarized. If wave propagates in the –ve z-direction the $E_s = E_o(a_x \pm i a_y) e^{+i\beta z}$

where in this case the +ve sign applies to right circular polarization and the '-ve' sign to left circular polarization

* From (iii) and (iv) the tip of the composite electric field as observed as a fixed point in the x-y plane moves along a circle as time progresses.

=> Circularly polarized waves can be produced by a helically wound wire antenna or by two linear sources that are oriented perpendicular to each other and fed with currents that are out of phase by 90°.

The locus of total field traces a circle can be seen if we examine the components at a point (say $z=0$).

$$E_x = E_0 \cos \omega t \quad , \quad E_y = E_0 \cos(\omega t + \pi/2) = -E_0 \sin \omega t$$

$$\therefore \quad |E|^2 = E_x^2 + E_y^2 = E_0^2$$ which is the equation of a circle. of radius E_0.

* Linear and circular polarizations are special cases of the more general case of the elliptical polarization.

=> An elliptically polarized wave is one in which the tip of the field traces an elliptic locus in a fixed transverse plane as the field changes with time.

=> Elliptical polarization is obtained when the x- and y- component are not same in magnitude $E_{0x} \neq E_{0y}$ and the phase difference b/w them is an odd multiple of $\pi/2$ - that is

$$\Delta\phi = \phi_y - \phi_x = \pm (2n+1)\pi/2, \quad n = 0, 1, 2, 3 \ldots$$

This allows the tip of the electric field to trace an ellipse in the x-y plane.

=> To represent that - when $z=0$ and $\Delta\phi = \phi_y - \phi_x = \pi/2$

so $E_x = E_{ox}\cos(\omega t) \rightarrow \cos\omega t = \frac{E_x}{E_{ox}}$

and $E_y = E_{oy}\cos(\omega t + \pi/2) = -E_0 \sin\omega t$

so $-\sin\omega t = \frac{E_y}{E_{oy}}$

Now $\cos^2\omega t + \sin^2\omega t = 1 \Rightarrow \frac{E_x^2}{E_{ox}^2} + \frac{E_y^2}{E_{oy}^2} = 1$

which is the eq. of an ellipse.. Notice that if $E_{ox} = E_{oy}$, we have circular polarization. Thus, it is special case of elliptical polarization. Also, we can show that linear polarization is a special case of elliptical polarization. Thus, the most general case is elliptical polarization.

Numericals - I Determine the polarization of a plane wave with

(i) $E(z,t) = 4e^{-0.25z}\cos(\omega t - 0.8z)\,a_x + 3e^{-0.25z}\sin(\omega t - 0.8z)\,a_y$ V/

(ii) $H(z,t) = H_0 e^{-j\beta z} a_x - 2H_0 e^{-j\beta z} a_y$

(iii) $E_s = E_0(a_x - j a_y)e^{-j\beta z}$

References:

- Elements of Electromagnetics, M N O Sadiku
- Elements of Electromagnetic Theory & Electrodynamics, Satya Prakash