An Introduction to Wavelets

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Wavelet analysis is a rapidly developing area in Mathematical Sciences. Morlet and Grossman were the first introduced the wavelets in the beginning of 1980. Based on the idea of wavelets as a family of functions constructed from translation and dilation of a single function ψ, called the 'mother wavelet', we define mother wavelet as [4]

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right) \; ; \; a, b \in \mathbb{R}, \; a \neq 0,$$

where a is called a scaling parameter, and b is a translation parameter which determines the location of the wavelet.

Definition (Wavelets)

A function ψ is a wavelet if it satisfies the following conditions

(i) $\psi(t) \to 0$ as $|t| \to \infty$

(ii) $\int_{-\infty}^{\infty} \psi(t)\, dt = 0$.

Definition

Wavelets in $L^2(\mathbb{R})$: A function $\psi \in L^2(\mathbb{R})$ is a wavelet if it satisfies the 'admissibility condition'

$$C_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi \ < \infty, \tag{1}$$

where $\hat{\psi}(\xi)$ is the Fourier transform of $\psi(t)$.

Example (The Haar Wavelet)

The Haar wavelet is one of the classic examples. It is defined by

$$\psi(t) = \begin{cases} 1, & \text{if } 0 \leq t < \frac{1}{2} \\ -1, & \text{if } \frac{1}{2} \leq t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

The Haar wavelet has compact support. It is obvious that

$$\int_{-\infty}^{\infty} \psi(t) dt = 0, \quad \text{and} \quad \int_{-\infty}^{\infty} |\psi(t)|^2 \, dt = 1.$$

This wavelet is well-localized in the time domain but it is not continuous.

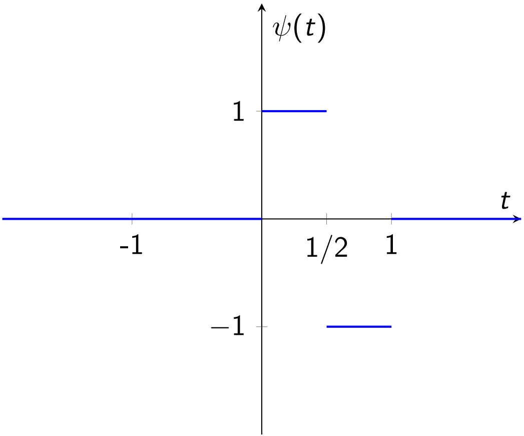

Figure: Haar Wavelet

Example

The function $\psi(t) = t\, e^{-t^2}$ satisfies the admissibility condition and hence wavelet.
Note that:

$$\int_{\mathbb{R}} |\psi(t)|^2 \, dt = \frac{\sqrt{\pi}}{4\sqrt{2}} < \infty,$$

$$\hat{\psi}(\xi) = -\frac{i\sqrt{\pi}\,\xi}{2}\, e^{-\xi^2/4},$$

and

$$C_\psi = \int_{\mathbb{R}} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi = \frac{\pi}{2} \; < \infty.$$

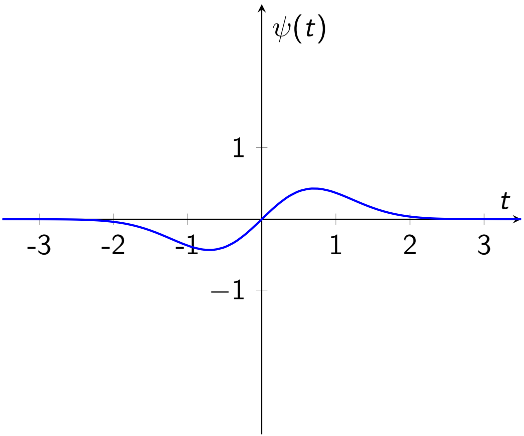

Figure: $\psi(t) = t\exp(-t^2)$

Example (The Mexican Hat wavelet)

The Mexican hat wavelet is defined by

$$\psi(t) = (1 - t^2)\exp(-\frac{t^2}{2}),$$

$$\hat{\psi}(\xi) = \sqrt{2\pi}\xi^2\exp(-\frac{\xi^2}{2}).$$

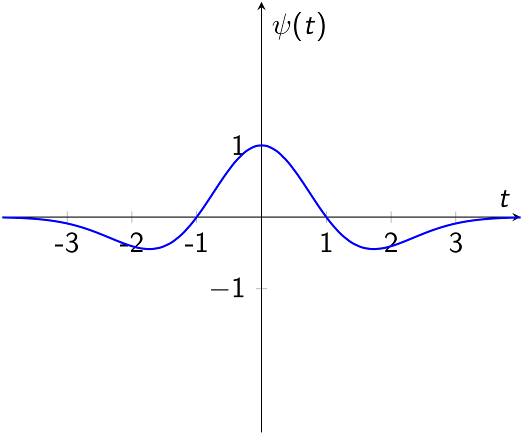

Figure: $\psi(t) = (1 - t^2)\exp(-\frac{t^2}{2})$

Example (The Shannon wavelet)

The Shannon function ψ whose Fourier transform satisfies

$$\hat{\psi}(\xi) = \chi_I(\xi),$$

where $I = [-2\pi, -\pi] \cup [\pi, 2\pi]$, is called the Shannon wavelet. The wavelet $\psi(t)$ is obtained from the inverse Fourier transform of $\hat{\psi}(\xi)$, so that

$$\psi(t) = \frac{\sin(\frac{\pi t}{2})}{(\frac{\pi t}{2})} \cos(\frac{3\pi t}{2}).$$

Example (The Morlet wavelet)

The Morlet wavelet is defined by

$$\psi(t) = \exp(i\xi_0 t - \frac{t^2}{2}),$$

$$\hat{\psi}(\xi) = \sqrt{2\pi} \exp(-\frac{(\xi-\xi_0)^2}{2}).$$

Theorem

If $\psi \in L^2(\mathbb{R})$ *is a wavelet and* ϕ *is an absolutely integrable function, then the convolution function* $(\psi * \phi)$ *is a wavelet.*

Proof.

$$
\begin{aligned}
\int_{-\infty}^{\infty} |(\psi * \phi)(x)|^2 \, dx &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \psi(x-u)\phi(u) du \right|^2 dx \\
&\leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\psi(x-u)| \, |\phi(u)| \, du \right)^2 dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\psi(x-u)| \, |\phi(u)|^{1/2} \, |\phi(u)|^{1/2} \, du \right)^2 dx \\
&\leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\psi(x-u)|^2 \, |\phi(u)| \, du \int_{-\infty}^{\infty} |\phi(u)| \, du \right) dx \\
&= \int_{-\infty}^{\infty} |\phi(u)| \, du \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi(x-u)|^2 \, |\phi(u)| \, dudx \\
&= \int_{-\infty}^{\infty} |\phi(u)| \, du \int_{-\infty}^{\infty} |\phi(u)| \int_{-\infty}^{\infty} |\psi(x-u)|^2 \, dxdu \\
&= \left(\int_{-\infty}^{\infty} |\phi(u)| \, du \right)^2 \|\psi\|_2^2 < \infty.
\end{aligned}
$$

Thus $\psi * \phi \in L^2(\mathbb{R})$. Moreover

$$\int_{-\infty}^{\infty} \frac{|\mathscr{F}(\psi * \phi)|^2}{|\xi|} d\xi = \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\xi)\hat{\phi}(\xi)\right|^2}{|\xi|} d\xi$$
$$= \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\xi)\right|^2 \left|\hat{\phi}(\xi)\right|^2}{|\xi|} d\xi$$
$$\leq \left(\sup \left|\hat{\phi}(\xi)\right|^2\right) \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\xi)\right|^2}{|\xi|} d\xi < \infty.$$

Thus the convolution function $\psi * \phi$ is a wavelet.

Example

If we take the Haar wavelet

$$\psi(t) = \begin{cases} 1, & \text{if } 0 \le t < \frac{1}{2} \\ -1, & \text{if } \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise,} \end{cases}$$

and convolute it with the following function

$$\phi(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

we obtain a simple wavelet

$$\begin{aligned}(\psi * \phi)(t) &= \int_{-\infty}^{\infty} \psi(t-u)\phi(u)du \\ &= \int_{-\infty}^{0} \psi(t-u)\phi(u)du + \int_{0}^{1} \psi(t-u)\phi(u)du \\ &+ \int_{1}^{\infty} \psi(t-u)\phi(u)du \\ &= \int_{0}^{1} \psi(t-u)du \\ &= \int_{t-1}^{t} \psi(x)dx.\end{aligned}$$

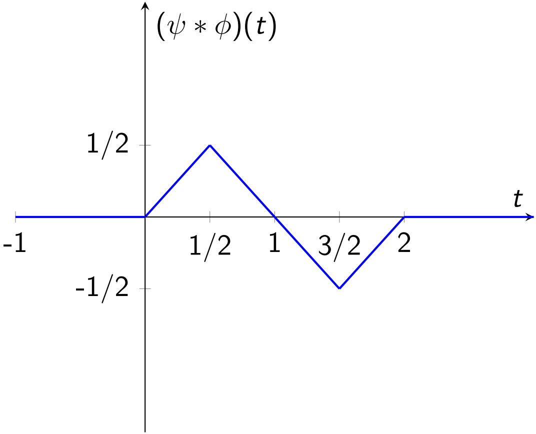

Figure: The wavelet $(\psi * \phi)(t)$

Example

The convolution of the Haar wavelet with $\phi(t) = \exp(-t^2)$ generates a smooth wavelet.

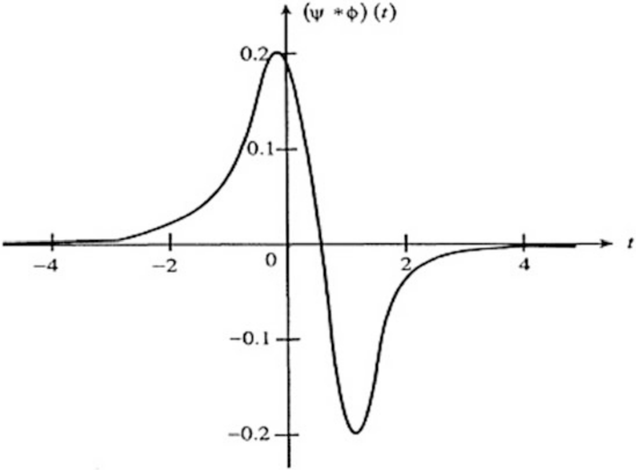

Figure: The wavelet $(\psi * \phi)(t)$

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