COMPLEX INTEGRATION

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SOME BASIC DEFINITIONS:

- **Path:** A continuous complex function γ defined on a closed and bounded interval $[a, b]$.

- **Closed Path:** The path is closed if $\gamma(a) = \gamma(b)$.

- **Simple Path:** The path is simple (Jordan arc) if it does not cross itself.

- **Multiply connected Path:** A closed path which intersects itself more than once.

- **Smooth Path:** If first order derivative at each point of the path exist and is continuous in its domain.
- **Piecewise Smooth Path:** If there exist a partition of $[a, b]$ and path is smooth on each subinterval.
- **Contour:** A curve consisting of a finite number of smooth arcs joined end to end.

PRIMITIVES

Suppose a function f is continuous in a domain D, then the following statements are equivalent:

I. f has antiderivative F in D.

II. The integrals of $f(z)$ along any path(lying entirely in D) between any two fixed points in D is independent of path.

I. The integral of $f(z)$ along every closed contour is zero.

CAUCHY'S THEOREM

Let $f(z)$ be analytic on and inside a simple closed contour C and let $f'(z)$ be also continuous on and inside C, then

$$\int_C f(z)dz = 0$$

CAUCHY-GOURSAT THEOREM

If a function $f(z)$ is analytic throughout a simply connected domain D, then for any simple closed contour C lying completely inside D, then

$$\int_C f(z)dz = 0$$

Note: In this theorem the condition of continuity of $f'(z)$ has been relaxed.

REMARK :

The integral of a function f (z) which is analytic throughout a simply connected domain D depends on the end points and not on the particular contour taken. Suppose α and β are inside D, C_1 and C_2 *are any contours inside D joining α to β, then*

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

THEOREM:

Suppose that γ *is a simple closed contour, described in the counterclock wise direction. Let* C_1, C_2, . . ., C_n *be positively oriented simple closed contours, where* C_1, C_2, . . ., C_n *are all inside* γ. *Interiors of all* C_1, C_2, . . ., C_n, *are disjoint. If a function* $f(z)$ *be analytic through out the closed region consisting of all points within and on* γ *except for the points interior to each* C_k,

then

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz$$

CAUCHY INTEGRAL FORMULA

Let $f(z)$ be analytic in a domain D and let γ be a simple closed contour in D, taken in the positive sense. Then

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(s)}{s-z}\,ds \;, where\; z \notin \gamma\,.$$

Example: With the help of Cauchy integral formula evaluate

$$\int_{\gamma} \frac{z^2-4z+4}{z+i}\,dz \;, where\; \gamma \; is\; the\; circle\; |z| = 2$$

Solution:

let $f(s) = s^2 - 4s + 4$ *and* $z = -i$,

we can easily see that $z = -i$ *lies inside the circle,*

also the function is analytic at all points within and on the contour γ,

Then by Cauchy's integral formula

$$\int_{\gamma} \frac{z^2 - 4z + 4}{z + i} dz = 2\pi i \, f(-i)$$

$$= 2\pi i(3 + 4i)$$

$$= \pi(6i - 8)$$ Answer

THANK YOU !