PHYS 3014: Statistical Mechanics

Lecture Notes
Part 4

Non- interacting Spin System

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Non-intracting Spin System

Consider an isolated system consisting of N non-interacting spins with spin $\frac{1}{2}$.
External magnetic field is B.
μ is the magnetic moment associated with each spin.

Since spins are located at different sites in solid. So we can distinguish them. A particular state or configuration of the system is specified by asigning the orientation up or down to all the N spins.

N = total no. of spins.

n = total no. of up spins i.e parallel to external magnetic field B.

$N - n$ = total no. of down spins i.e. anti-parallel to B.

Total energy of the system

$$E = n(-\mu B) + (N-n)\mu B$$

$$= -(2n - N)\mu B$$

$$\Rightarrow \quad n = \frac{N}{2} - \frac{E}{2\mu B}$$ for a given set of N and E

The no. of microstates

$$\Omega(n, N) = \frac{\lfloor N}{\lfloor n \, \lfloor N-n}$$

Entropy

$$S = k \ln \Omega$$

Temperature T is given by

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{\partial S}{\partial n}\frac{\partial n}{\partial E} = -\frac{1}{2\mu B}\frac{\partial S}{\partial n}$$

$$\ln \Omega = N \ln N - N - n \ln n + n - (N-n)\ln(N-n) + N - n$$

$$\therefore \quad S = kN \ln N - kn \ln n - k(N-n)\ln(N-n)$$

$$\therefore \quad \frac{1}{T} = -\frac{k}{2\mu B}\left[-\ln n - 1 + \ln(N-n) + 1\right]$$

$$= -\frac{k}{2\mu B}\ln\left(\frac{N-n}{n}\right)$$

or, $$\frac{2\mu B}{kT} = \ln\left(\frac{n}{N-n}\right)$$

or, $$\frac{N-n}{n} = \exp\left(\frac{-2\mu B}{kT}\right)$$

or, $$\frac{n}{N} = \frac{1}{1+\exp\left(\frac{-2\mu B}{kT}\right)}$$

or, $$n = \frac{N}{1+\exp\left(\frac{-2\mu B}{kT}\right)}$$

The probability that a given spin is up

$$= \frac{n}{N}$$

$$= \frac{\exp\left(\frac{\mu B}{KT}\right)}{\exp\left(\frac{\mu B}{KT}\right) + \exp\left(-\frac{\mu B}{KT}\right)}$$

Energy of the system having particular configuration is

$$E = -\mu B\left[\frac{2N}{1+\exp\left(\frac{-2\mu B}{KT}\right)} - N\right]$$

$$= -N\mu B\left[\frac{1-\exp\left(\frac{-2\mu B}{KT}\right)}{1+\exp\left(\frac{-2\mu B}{KT}\right)}\right]$$

$$E = -N\mu B \tanh\left(\frac{\mu B}{KT}\right)$$

Total Magnetization of the system having particular configuration is

$$M = (2n - N)\mu = \frac{2N}{1+\exp\left(\frac{-2\mu B}{KT}\right)} - N = \mu N \tanh\left(\frac{\mu B}{KT}\right)$$

Two level System with degenerate energy level

Consider a simple two energy levels system.

Particles can be in any energy state.

doubly degenerate ——— ϵ

non-degenerate ——— 0

The ground state is non-degenerate and has zero energy.

The excited state has energy ϵ and is doubly degenerate.

N is the no. of non-interacting particles.

N_e is the no. of particles occupying in the higher energy state of energy ϵ.

Total energy of system $E = N_e \epsilon$

The total no. of ways that N_e could be selected out of N is

$$\frac{N!}{N_e!\,(N-N_e)!}$$

N_e particles are distributed between two degenerate excited energy levels. Each particle has two choice. The total no. of ways to distribute N_e particles in two degenerate energy levels is 2^{N_e}.

$\therefore$ The no. of accessible microstates is

$$\Omega = 2^{N_e} \cdot \frac{N!}{N_e!\,(N-N_e)!}$$

Entropy of the system

$$S = k \ln \Omega = k\left[\ln N! + N_e \ln 2 - \ln N_e! - \ln (N-N_e)!\right]$$

or, $$S = K\Big[N \ln N - N + N_e \ln 2 - N_e \ln N_e + N_e - (N-N_e)\ln(N-N_e) + (N-N_e)\Big]$$

$$= K\left[N \ln N - N_e \ln\left(\frac{N_e}{2}\right) - (N-N_e)\ln(N-N_e)\right]$$

$$= -NK\left[\frac{N_e}{N}\ln\left(\frac{N_e}{2N}\right) + \left(1-\frac{N_e}{N}\right)\ln\left(1-\frac{N_e}{N}\right)\right]$$

Now $$N_e = \frac{E}{\epsilon}$$

$$\therefore\ S = -NK\left[\left(\frac{E}{N\epsilon}\right)\ln\left(\frac{E}{2N\epsilon}\right) + \left(1-\frac{E}{N\epsilon}\right)\ln\left(1-\frac{E}{N\epsilon}\right)\right]$$

we know that $$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N$$

After rearrangement we can get

$$E = \frac{2N\epsilon}{e^{\frac{\epsilon}{KT}} + 2}$$

References:

- Statistical Mechanics by R. K. Pathria
- Statistical Mechanics by B. K. Agarwal and M. Eisner
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- Elementary Statistical Physics by C. Kittel
- Fundamentals of Statistical and Thermal Physics by F. Reif
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Thank You

For any questions/doubts/suggestions and submission of assignments

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