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Topics: Propagation of Electromagnetic Waves in Ionized gas

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A simple Model for Dynamical Conductivity

If the field frequency developed across a conductor is raised, the conducting electrons due to their inertia, follow the field with raising difficulty. It suggests a decrease in the conductivity with increasing frequency. To understand this, let us suppose the simple model given by Drude. According to this model a metal contains a certain number (say) n_0 of electrons per unit volume free to move under the action of applied electric fields; but subject to damping force due to collision.

The collisions occur between electrons and Lattice vibrations, Lattice imperfections, and impurities.

If b is the damping constant, then the damping force may be written as -

$$F_{damp} = -bmv \quad \text{-------} \quad (i)$$

If E is electric field applied across a conductor, then from Newton's 2nd law, the eq. of motion of conducting electron -

$$m\frac{dv}{dt} = eE - bmv \quad \text{------} \quad (ii)$$

For rapidly oscillating fields, the displacement of the electron is small compared to a wavelength, which is $E = E_0 e^{-i\omega t}$

Now from eq. (ii) $\quad m\frac{dv}{dt} = eE_0 e^{-i\omega t} - bmv \quad - \quad (iii)$

Here E_0 is the electric field at the average position of the electron

Now, the current density J -

$$J = n_0 e v \quad \text{or} \quad v = \frac{J}{n_0 e}$$

from eq. (iii)

$$m\frac{d}{dt}\left(\frac{J}{n_0 e}\right) = eE_0 e^{-i\omega t} - bm\left(\frac{J}{n_0 e}\right) \quad \text{-----} \quad (iv)$$

Simplifying and rearranging this eq.

$$m\frac{dJ}{dt} + bmJ = n_0 e^2 E e^{-i\omega t}$$

Also for time varying current density $J = J_0 e^{-i\omega t}$

Now from eq. (iv) $m\frac{d}{dt}\left(J_0 e^{-i\omega t}\right) + bmJ_0 e^{-i\omega t} = n_0 e^2 E_0 e^{-i\omega t}$

$$mJ_0(-i\omega)e^{-i\omega t} + bmJ_0 e^{-i\omega t} = n_0 e^2 E_0 e^{-i\omega t}$$

$$-im\omega J + bmJ = n_0 e^2 E$$

$$\Downarrow$$

$$J = \frac{n_0 e^2 E}{m(b-i\omega)} \quad \text{-----} \quad (v)$$

If we compare this with $J = \sigma E$ – we find that –

$$\boxed{\sigma = \frac{n_0 e^2}{m(b-i\omega)}} \quad \text{-----} \quad (vi)$$

In a metal such as a copper where $n_0 \simeq 1\times10^{28}$ electrons/m^3, $\sigma = 5\times10^7$ S/m has an empirical damping constants $b \simeq 3\times10^{13}\ sec^{-1}$.

→ It is clear that for frequencies of the order of or smaller than, microwave frequencies ($\sim 10^{10}\ sec^{-1}$) the electrical conductivity is essentially real (current is in phase with the applied field) and it is independent of frequency ($\omega \simeq 0$) – and thus it takes the form $\sigma = \frac{n_0 e^2}{mb}$ ----- (vii)

This is well known Lorentz-Drude expression for conductivity

* At higher frequencies, however, the conductivity is complex and depends markedly on frequency in a manner defined in equat^n (vi).

* Maxwell's equations and eq. of EM waves in ionised me

In some of cases of ionised gases when the pressure is quit low such as the ionosphere or a plasma, we may suppose that there a no collisions and hence no energy losses (damping constant $b=0$) so that the conductivity σ defined from eq. (VI) becomes purely imaginary and it is thus given by $\sigma = -\frac{n_0 e^2}{mi\omega} \simeq \frac{i n_0 e^2}{m\omega}$.

Now the differential form of Maxwell's eqs.

$$\left.\begin{aligned} &\nabla\cdot D = 0 \\ &\nabla\cdot B = 0 \\ &\nabla\times E = -\frac{\partial B}{\partial t} \\ &\nabla\times H = J + \frac{\partial D}{\partial t} \end{aligned}\right\}$$

with $J = \sigma E$ and $\sigma = \frac{i n_0 e^2}{m\omega}$

$B = \mu H$

$D = \varepsilon E$ $\quad \varepsilon = \varepsilon_0$

$\rho_v = 0$ $\quad \mu = \mu_0$

$$\nabla\times E = -\mu_0 \frac{\partial H}{\partial t} \quad -(a) \qquad \nabla\cdot E = 0 \quad -(c)$$

$$\nabla\times H = \sigma E + \varepsilon_0 \frac{\partial E}{\partial t} \quad -(b) \qquad \nabla\cdot H = 0 \quad -(d)$$

→ Now taking curl of eq. (a)

$$\nabla\times\nabla\times E = -\mu_0 \frac{\partial}{\partial t}(\nabla\times H)$$

$$(\nabla\cdot E)\nabla - (\nabla^2 E) = -\mu_0 \frac{\partial}{\partial t}\left(\sigma E + \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\nabla^2 E - \mu_0\sigma \frac{\partial E}{\partial t} - \mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2} = 0 \quad - (\text{VIII})$$

Similarly taking curl of eq. (b).

and we get

$$\nabla^2 H - \mu_0\sigma \frac{\partial H}{\partial t} - \mu_0\varepsilon_0 \frac{\partial^2 H}{\partial t^2} = 0 \quad - (\text{VIII})$$

Eqs. (VIII) and (IX) represent wave eqs. in terms of electro-magnetic field vectors E and H in ionised medium.

These equations are vector equations of similar form, therefore each of six components of E and H separately satisfies the same scalar wave equation of the form-

$$\nabla^2 f - \mu_0 \sigma \frac{\partial f}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 f}{\partial t^2} = 0 \quad - \quad (X)$$

where f is a scalar and can stand for any of six components of E and H.

→ The plane wave solutions of eqs those are mentioned above.

$$\left.\begin{aligned} E &= E_0 e^{ik\cdot r - i\omega t} \\ H &= H_0 e^{ik\cdot r - i\omega t} \\ f &= f_0 e^{ik\cdot r - i\omega t} \end{aligned}\right\} \quad - \quad (XI)$$

where E_0 and H_0, f_0 are complex amplitudes which are constant in space and time, and k is a vector quantity known as a wave vector or wave propagation vector and defined as

$$K = k\cdot\hat{n} = \frac{2\pi}{\lambda}\cdot\hat{n} = \frac{\omega}{v}\cdot\hat{n}$$

here $\hat{n}$ is a unit vector along k and v is phase velocity of the wave. , Now from eqs (XI) $\nabla^2 f = -K^2 f$

and $\frac{\partial f}{\partial t} = -i\omega f$ and $\frac{\partial^2 f}{\partial t^2} = -\omega^2 f$

Now putting these into eq. (X)

$$-K^2 f + i\omega\mu_0\sigma f + \mu_0\varepsilon_0\omega^2 f = 0$$

$$(K^2 - i\omega\mu_0\sigma - \mu_0\varepsilon_0\omega^2) f = 0$$

As f is an arbitrary component of field vector, hence above eq. holds only if $K^2 - i\omega\mu_0\sigma - \mu_0\varepsilon_0\omega^2 = 0$

$$K^2 = \mu_0\varepsilon_0\omega^2\left[1 + \frac{i\sigma}{\omega\varepsilon_0}\right] \quad -$$

Substituting the value of σ in Last equation.

$$K^2 = \mu_0 \varepsilon_0 \omega^2 \left[1 - \frac{n_0 e^2}{m \varepsilon_0 \omega^2}\right] = \frac{\omega^2}{c^2}\left[1 - \frac{n_0 e^2}{m \varepsilon_0 \omega^2}\right]$$

$$K^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_p^2}{\omega^2}\right] \quad \text{where } \omega_p^2 = \frac{n_0 e^2}{m \varepsilon_0}$$

$-(\text{XII})$

ω_p is known as the plasma frequency. and $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

* If n is the refractive index, then $n = \frac{c}{v}$

$\therefore K = \frac{\omega}{v} = \frac{n\omega}{v}$

$K^2 = \frac{n^2 \omega^2}{c^2}$ from equation $(\text{x}11)$.

the refractive index of a plasma medium-

$$\boxed{n^2 = 1 - \frac{\omega_p^2}{\omega^2}}$$

* For high frequency region $\omega > \omega_p$, the refractive index n $\left(n = \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}\right)$ is real and therefore waves propagate fre

* For low frequency region $\omega < \omega_p$, the refractive index, n is purely imaginary. As a result such EM waves incident on a plasm will be reflected from the surface.

$\therefore \quad K = \frac{n\omega}{c}$

if we replace in in place of n in above eq

$K = \frac{in\omega}{c}$

Now, the sols for E and H for low frequency regime may be expressed as $E = E_0 e^{ik(\hat{n}\cdot r) - i\omega t}$

$$E = E_0 e^{-(\omega n/c)(\hat{n}\cdot r)} e^{-i\omega t} \quad - (\text{XIII})$$

and $$H = H_0 e^{-\omega n/c (\hat{n}\cdot r)} e^{-i\omega t} \quad - (\text{XIV})$$

These equations represent that within the electromagnetic field vectors E and H will fall off exponentially with distance from the surface.

* The skin depth or penetration depth for the plasma

The skin depth $\delta_{plasma} = \frac{1}{\beta} = \frac{1}{\left(\frac{\omega n}{c}\right)} \frac{1}{\left[\frac{\omega_p^2}{\omega^2} - 1\right]^{1/2}}$

$\because \beta = (\omega n/c)$

$\therefore \delta_{plasma} = \frac{c}{\sqrt{\omega_p^2 - \omega^2}} = \left(\frac{c}{\omega_p}\right)$ $(\because \omega << \omega_p)$

⇒ At the laboratory scale plasma densities are in the range $n_0 \simeq 10^{10} - 10^{12}$ electrons/m^3. This means that plasma frequency $\omega_p \simeq 6\times10^{10} - 6\times10^{12}$ sec^{-1}), so that the typical penetration depths are of the order of 0.5 cm to 5×10^{-3} cm for low frequency fields.

* Critical frequency for propagation of EM waves in plasma

→ we know that the transmission is possible in plasma only when the refractive index is real.

The refractive index n is given by $n^2 = 1 - \frac{\omega_p^2}{\omega^2}$

or $n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$ where $\omega_p^2 = \frac{n_0 e^2}{m \epsilon_0}$

If refractive index n is real, then $\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \geq 0$

or $\omega_0^2 \geq \omega_p^2$ if ω_0 is the critical angular frequency for propagation of EM waves in plasma then $\omega_0 = \omega_p = \sqrt{\frac{n_0 e^2}{m \varepsilon_0}}$ and thus critical frequency

$$F_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{n_0 e^2}{m \varepsilon_0}} \Rightarrow \boxed{F_0 = \frac{1}{2\pi}\sqrt{\frac{n_0 e^2}{m \varepsilon_0}}}$$

$$F_0 = \frac{1}{2 \times 3.14}\sqrt{\frac{n_0 \times (1.6 \times 10^{-19})^2}{(9 \times 10^{-31} \times 8.85 \times 10^{-12})}} = 9\sqrt{n_0}$$

* $\boxed{F_0 = 9\sqrt{n_0}}$

Numerical-1 – calculate the plasma frequency and maximum – penetration depth for a plasma containing 10^{18} electrons/m^3.

Solution – Plasma frequency $F_0 = 9\sqrt{n_0}$

$$= 9 \times \sqrt{10^{18}} = 9 \times 10^9 \text{ Hz}$$

F_0 = 9000 MHz Ans.

References:

- Elements of Electromagnetics, M N O Sadiku
- Elements of Electromagnetic Theory & Electrodynamics, Satya Prakash