The Continuous Wavelet Transform

Dr. Sunil Kumar Singh

Associate Professor
Department of Mathematics
Mahatma Gandhi Central University
Motihari-845401, Bihar, India

E-mail: sks_math@yahoo.com

April 06, 2020

Definition

The continuous wavelet transform of a function ϕ with respect to the wavelet ψ is defined by

$$(W_\psi \phi)(a,b) = \langle \phi, \psi_{a,b} \rangle = \int_{-\infty}^{\infty} \phi(t)\, \bar{\psi}_{a,b}(t)\, dt, \tag{1}$$

where $\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi(\frac{t-b}{a}), a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$, provided the integral exists.

If $\phi \in L^2(\mathbb{R})$ and $\psi \in L^2(\mathbb{R})$, then using the Parseval's formula for Fourier transform, (1) can be re-written in the form:

$$\begin{aligned}(W_\psi \phi)(a,b) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(\xi) \sqrt{|a|}\ \overline{\hat{\psi}(a\xi)}\ e^{ib\xi} d\xi \\ &= \frac{\sqrt{|a|}}{2\pi} \int_{-\infty}^{\infty} e^{ib\xi} \hat{\phi}(\xi)\ \overline{\hat{\psi}(a\xi)} d\xi \\ &= \mathscr{F}^{-1}\left[\sqrt{|a|}\hat{\phi}(\xi)\ \overline{\hat{\psi}(a\xi)}\right](b)\end{aligned} \tag{2}$$

where $\hat{\phi}$ denotes Fourier transform of ϕ, similarly $\hat{\psi}$.

Example

The wavelet transform of a constant function is zero.

Basic properties of wavelet transforms I

If ψ and ϕ are wavelets and f, g are functions which belong to $L^2(\mathbb{R})$, then

(i) (Linearity)

$$(W_\psi(\alpha f + \beta g))(a, b) = \alpha(W_\psi f)(a, b) + \beta(W_\psi g)(a, b), \qquad (3)$$

for any scalars α, β.

(ii) (Translation)

$$(W_\psi T_c f)(a, b) = (W_\psi f)(a, b - c), \qquad (4)$$

where T_c is the translation operator defined by $T_c f(t) = f(t - c)$.

(iii) (Dilation)

$$(W_\psi(D_c f)(a, b) = (W_\psi f)\left(\frac{a}{c}, \frac{b}{c}\right), c > 0, \qquad (5)$$

where D_c is the dilation operator defined by $D_c f(t) = \frac{1}{\sqrt{|c|}} f(\frac{t}{c}), c > 0$.

(iv) (Symmetry)

$$(W_\psi \phi)(a,b) = \overline{(W_\phi \psi)\left(\frac{1}{a}, -\frac{b}{a}\right)}, a \neq 0, \tag{6}$$

where D_c is the dilation operator defined by $D_c f(t) = \frac{1}{\sqrt{|c|}} f(\frac{t}{c}), c > 0$.

(v) (Parity)

$$(W_{P\psi} Pf)(a,b) = (W_\psi f)(a,-b), \tag{7}$$

where P is the Parity operator defined by $Pf(t) = f(-t)$.

(vi) (Antilinearity)

$$(W_{\alpha\psi+\beta\phi} f)(a,b) = \overline{\alpha}(W_\psi f)(a,b) + \overline{\beta}(W_\phi f)(a,b), \tag{8}$$

for any scalars α, β.

(vii)

$$(W_{T_c\psi} f)(a,b) = (W_\psi f)(a, b+ca). \tag{9}$$

(viii)

$$(W_{D_c\psi} f)(a,b) = (W_\psi f)(ca, b), c > 0. \tag{10}$$

Theorem (Parseval's formula)

Assume that $C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} \, d\xi < \infty$. *Let* $f, g \in L^2(\mathbb{R})$ *and* $(W_\psi f)(a,b), (W_\psi g)(a,b)$ *be the wavelet transforms of* f, g *respectively, with respect to the wavelet* $\psi \in L^2(\mathbb{R})$. *Then*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)(a,b) \overline{(W_\psi g)(a,b)} \frac{da db}{a^2} = C_\psi \int_{-\infty}^{\infty} f(t) \overline{g(t)} dt.$$

Proof.

We have by definition of wavelet transform,

$$
\begin{aligned}
(W_\psi f)(a,b) &= \int_{-\infty}^{\infty} f(t)\, \overline{\psi_{a,b}(t)} dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \sqrt{|a|}\, \overline{\hat{\psi}(a\xi)}\, e^{ib\xi} d\xi.
\end{aligned} \tag{11}
$$

Similarly,

$$
\overline{(W_\psi g)(a,b)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{g}(\omega)} \sqrt{|a|}\, \hat{\psi}(a\omega)\, e^{-ib\omega} d\omega. \tag{12}
$$

□

Therefore,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(W_\psi f)(a,b)\overline{(W_\psi g)(a,b)}\frac{dadb}{a^2}$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dadb}{a^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|a|\hat{f}(\xi)\overline{\hat{g}(\omega)}\ \overline{\hat{\psi}(a\xi)}\ \hat{\psi}(a\omega)e^{ib(\xi-\omega)}d\xi d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{da}{|a|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{g}(\omega)}\ \overline{\hat{\psi}(a\xi)}\ \hat{\psi}(a\omega)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ib(\xi-\omega)}db\right)d\xi d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{da}{|a|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{g}(\omega)}\ \overline{\hat{\psi}(a\xi)}\ \hat{\psi}(a\omega)\delta(\xi-\omega)d\xi d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{da}{|a|}\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{g}(\xi)}\ \left|\hat{\psi}(a\xi)\right|^2\ d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{g}(\xi)}\ \left(\int_{-\infty}^{\infty}\frac{\left|\hat{\psi}(a\xi)\right|^2}{|a|}\ da\right)\ d\xi$$

$$= C_\psi\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{g}(\xi)}\ d\xi = C_\psi\int_{-\infty}^{\infty}f(t)\overline{g(t)}\ dt.$$

Theorem (Inversion formula)

If $f \in L^2(\mathbb{R})$ *and* $0 < C_\psi < \infty$, *then* f *can be reconstructed by the formula*

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} (W_\psi f)(a, b)\, \psi_{a,b}(t)\, \frac{dadb}{a^2}. \tag{13}$$

Proof.

For any $f, g \in L^2(\mathbb{R})$, we have by Parseval's formula for wavelet transforms

$$
\begin{aligned}
C_\psi \langle f, g\rangle &= C_\psi \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (W_\psi f)(a,b)\overline{(W_\psi g)(a,b)}\,\frac{dadb}{a^2} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (W_\psi f)(a,b)\overline{\int_{-\infty}^{\infty} g(t)\ \overline{\psi_{a,b}(t)}dt}\ \frac{dadb}{a^2} \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (W_\psi f)(a,b)\ \psi_{a,b}(t)\ \frac{dadb}{a^2}\right)\overline{g(t)}dt \\
&= \left\langle \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (W_\psi f)(a,b)\ \psi_{a,b}(t)\ \frac{dadb}{a^2}, g(t)\right\rangle
\end{aligned}
$$

□

which implies that

$$C_\psi f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(W_\psi f)(a,b)\ \psi_{a,b}(t)\ \frac{dadb}{a^2}$$
$$f = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(W_\psi f)(a,b)\ \psi_{a,b}(t)\ \frac{dadb}{a^2}.$$

If $f = g$, then we have by Parseval's formula

$$C_\psi \left\|f\right\|^2 = \int_{-\infty}^{\infty}\left|(W_\psi f)(a,b)\right|^2 \frac{dadb}{a^2}$$

References I

[1] A. Boggess, and F. J. Narcowich, A First Course in Wavelets with Fourier Analysis, Wiley; 2nd ed., 2009.

[2] C. K. Chui : *An Introduction to Wavelets*, Academic Press, New York, 1992.

[3] I. Daubechies : *Ten Lectures on Wavelets*, CBMSNSF Regional Conference Series in Applied Mathematics (SIAM,1992).

[4] Lokenath Debnath and Firdous Ahmad Shah : *Wavelet Transforms and Their Applications*, 2nd ed., Brkhäuser, Boston, 2015.

[5] E. Hernáddez and G. Weiss : *A First Course on Wavelets*, CRC Press, Boca Raton, Florida, 1996.

[6] T. H. Koornwinder : *Wavelets*, World Scientific Publishing Co.Pvt.Ltd., Singapore , 1993.

[7] R. S. Pathak : *The Wavelet transform*, Atlantis Press/ world Scientific, France, 2009.

References II

[8] P. Wojtaszczyk : *A Mathematical Introduction to Wavelets*, Cambridge University Press, 1997.

[9] D. F Walnut : *An Introduction to Wavelet Analysis*, Brkhäuser, 2004.

[10] A. H. Zemanian : *Generalized Integral Transformations*, Interscience Publishers, New York, 1996.

THANK YOU