

Applications to Quantum Statistics (B-E Statistics)



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- In this lecture, we will discuss the concept of photon gas and use B-E statistics to an assembly of photons and derive the Planck's law of black body radiation.
- Moreover, we will also discuss the behavior of an assembly of ideal bosons (strongly degenerate) at low temperatures and will come to know about one of the most remarkable quantum effects: **Bose-Einstein Condensation** phenomenon.
- This phenomenon was used to understand the peculiar properties of ^4He .

Planck's law of black body radiation

- Consider radiation as an assembly of photons in a chamber. As photons are bosons, having zero rest mass and do not obey Pauli's exclusion principle. Moreover, their number is not conserved so that $A(e^\alpha=1; \alpha=0)$. Such a system has to be treated as completely degenerate.
- Let us consider an assembly of photons inside a cavity in a heat reservoir at temperature T . Photons are continuously being emitted and absorbed by the walls of the container so that their number is not conserved however, their energy remains constant.
- Some properties of Photons are :
 - Photons are relativistic particles with zero rest mass.
 - Photons are indistinguishable and their chemical potential is zero.
 - Photons are Bose particles with integral spin having two states of polarization (clockwise and anti-clockwise).
- Volume of each allowed eigen state in the phase space is h^3 .

- Total number of eigen states in the momentum range p to $(p+dp)$ is given by –

$$g(p) dp = \frac{4\pi p^2 dp}{h^3 / V} = \frac{4\pi V p^2 dp}{h^3} \dots\dots(1)$$

For a photon,

$$p = \frac{h\nu}{c} \text{ or } dp = \frac{h d\nu}{c}$$

Now, the total number of eigen states between frequencies ν and $(\nu+d\nu)$ is given by –

$$g(\nu) d\nu = \frac{4\pi V \nu^2 d\nu}{c^3} \dots\dots(2)$$

since, there are two modes of propagation for each photon, therefore total number of eigen states between frequencies ν and $(\nu+d\nu)$ is given by –

$$g(\nu) d\nu = \frac{8\pi V \nu^2 d\nu}{c^3} \dots\dots(3)$$

Applying Bose-Einstein distribution law, we get

$$n(\nu) d\nu = \frac{g(\nu) d\nu}{e^{\alpha + \beta \epsilon} - 1} \dots\dots(4)$$

Here, $\alpha=0$, $\beta=1/kT$ and $\epsilon=h\nu$; above eqn. can be rewritten as –

$$n(\nu) d\nu = 8\pi V \frac{\nu^2}{c^3} \frac{d\nu}{e^{h\nu/kT} - 1}$$
$$\frac{n(\nu) d\nu}{V} = \frac{8\pi \nu^2}{c^3} \frac{d\nu}{e^{h\nu/kT} - 1} \dots\dots\dots(5)$$

Since each photon has energy $h\nu$, the energy density $u(\nu)d\nu$ defined as the amount of energy per unit volume lying between the frequencies ν and $(\nu+d\nu)$ is -

$$u(\nu) d\nu = \frac{h\nu n(\nu) d\nu}{V} = \frac{8\pi h \nu^3}{c^3} \frac{d\nu}{e^{h\nu/kT} - 1} \dots\dots(6)$$

This is well known Planck's law of radiation in terms of frequency.

In terms of wavelength, λ

$$u(\lambda) d\lambda = \frac{8\pi hc}{\lambda^5} \frac{-d\lambda}{e^{hc/\lambda kT} - 1}$$

$$|u(\lambda) d\lambda| = u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda kT} - 1} \dots\dots(7)$$

This is well known Planck's law of radiation in terms of wavelength.

Rayleigh-Jeans law: derivation from Planck's law

For long wavelengths,

$$\frac{hc}{\lambda kT} < 1 \quad \text{and} \quad e^{hc/\lambda kT} = 1 + \frac{hc}{\lambda kT} \quad (\text{approx.})$$

$$u_{\lambda} d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda \quad \dots\dots(8)$$

Wien's displacement law: derivation from Planck's law

differentiating Planck's law;

$$\frac{du_{\lambda}}{d\lambda} = 0 \quad \Rightarrow \quad e^{hc/\lambda kT} \left[1 - \frac{hc}{5\lambda kT} \right] = 1$$

$$\text{or} \quad e^x \left[1 - \frac{x}{5} \right] = 1 \quad \text{where} \quad x = \frac{hc}{\lambda kT}$$

Roots of the eqⁿ. are –

$$x = 0 \quad \text{and} \quad x = 5 - \frac{5}{e^x} \Rightarrow x = 4.965 \quad (\text{but } x \neq 0)$$

$$\therefore x = \frac{hc}{\lambda kT} = 4.965$$

or,

$$\lambda T = b = \text{a constant } (2.898 \times 10^{-3} \text{m-K})$$

This is known as **Wien's displacement law**.

Strongly Degenerate Boson Gas: B-E Condensation

Consider a perfect Bose-Einstein gas of n bosons. Let these particles be distributed among different quantum states such that there are n_1, n_2, \dots number of particles in quantum states having energies $\varepsilon_1, \varepsilon_2, \dots$ respectively.

Number of particles lying between energy range ε and $(\varepsilon+d\varepsilon)$ is given by -

$$n(\varepsilon)d\varepsilon = \frac{4\sqrt{2}\pi V m^{3/2}}{h^3} \frac{\varepsilon^{1/2} d\varepsilon}{e^{\alpha+\varepsilon/kT} - 1} \quad \dots(i)$$

let us put $x = \frac{\varepsilon}{kT}$ or $dx = \frac{d\varepsilon}{kT}$

Eqⁿ. (i) becomes

$$n(\varepsilon)d\varepsilon = \frac{V}{h^3} \left(2\pi mkT\right)^{3/2} \frac{2}{\sqrt{\pi}} \frac{x^{1/2} dx}{e^{\alpha+x} - 1} \dots(ii)$$

Now, total number of particles is given by –

$$N = \int_0^{\infty} n(\varepsilon)d\varepsilon = \frac{V}{h^3} \left(2\pi mkT\right)^{3/2} \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{x^{1/2} dx}{e^{\alpha+x} - 1}$$

$$N = \frac{V}{h^3} \left(2\pi mkT\right)^{3/2} f_1(\alpha) \dots(iii)$$

$$\text{where } f_1(\alpha) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{x^{1/2} dx}{e^{\alpha+x} - 1} \dots(iv)$$

Total energy of the system is given by –

$$E = \int_0^{\infty} \varepsilon n(\varepsilon) d\varepsilon = \frac{V}{h^3} \left(2\pi m k T\right)^{3/2} \cdot kT \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{x^{3/2} dx}{e^{\alpha+x} - 1}$$

$$E = \frac{3V}{2h^3} \left(2\pi m k T\right)^{3/2} \cdot kT f_2(\alpha) \quad \dots\dots(v)$$

$$\text{where } f_2(\alpha) = \frac{4}{3\sqrt{\pi}} \int_0^{\infty} \frac{x^{3/2} dx}{e^{\alpha+x} - 1} \quad \dots\dots(vi)$$

According to B-E distribution law, the most probable distribution is given by –

$$n_i = \frac{g_i}{e^{\alpha+\beta\varepsilon_i} - 1} = \frac{g_i}{\frac{1}{A} e^{\beta\varepsilon_i} - 1}$$

As the number of particles in a state can't be negative, i.e. $n_i \geq 0$ or

$$e^{\alpha+\beta\varepsilon_i} = \frac{1}{A} e^{\beta\varepsilon_i} \geq 1$$

So, the integral functions $f_1(\alpha)$ and $f_2(\alpha)$ can be evaluated as –

$$f_1(\alpha) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{x^{1/2} dx}{e^{\alpha+x} - 1} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} x^{1/2} A e^{-x} \left(1 - A e^{-x}\right)^{-1} dx$$

After expanding above and solving it, we get

$$f_1(\alpha) = A + \frac{A^2}{2^{3/2}} + \frac{A^3}{3^{3/2}} + \dots = \sum_{r=1}^{\infty} \frac{A^r}{r^{3/2}} \quad \dots\dots(vii)$$

Similarly,

$$f_2(\alpha) = A + \frac{A^2}{2^{5/2}} + \frac{A^3}{3^{5/2}} + \dots = \sum_{r=1}^{\infty} \frac{A^r}{r^{5/2}} \quad \dots\dots(viii)$$

Now. eqⁿ. (iii) and (v) becomes

$$N = \frac{V}{h^3} (2\pi mkT)^{3/2} \sum_{r=1}^{\infty} \frac{A^r}{r^{3/2}}$$

$$N = \frac{V}{h^3} (2\pi mkT)^{3/2} \left(A + \frac{A^2}{2^{3/2}} + \dots \right) \dots\dots(ix)$$

$$E = \frac{3V}{2h^3} (2\pi mkT)^{3/2} .kT \sum_{r=1}^{\infty} \frac{A^r}{r^{5/2}}$$

$$E = \frac{3V}{2h^3} (2\pi mkT)^{3/2} .kT \left(A + \frac{A^2}{2^{5/2}} + \dots \right) \dots\dots(x)$$

again,

$$\frac{E}{N} = \frac{3}{2} kT \left(A + \frac{A^2}{2^{5/2}} + \dots \right) \left(A + \frac{A^2}{2^{3/2}} + \dots \right)^{-1}$$

on further solving, we get

$$E = \frac{3}{2} NkT \left(1 - \frac{A}{2^{3/2}} + \frac{A}{2^{5/2}} \dots \right) \dots\dots(xi)$$

Thus, value of

$$f_1(\alpha) = \frac{Nh^3}{v(2\pi mkT)^{3/2}} \dots\dots(xii)$$

For $A \ll 1$, $f_1(\alpha) = A = e^{-\alpha} = \frac{Nh^3}{v(2\pi mkT)^{3/2}} \dots\dots(xiii)$

If the density of particles is increased and/or temperature is decreased, the value of A increases.

The parameter A is called degeneracy parameter and criterion of degeneracy is based on $\frac{N/V}{(mT)^{3/2}}$

So, the degree of degeneracy will be large when temperature is low, particle density is large and mass of each boson is small.

Thus, for low energy values, limiting case of the highest degeneracy in B-E gas is when $A=1$, $\alpha=0$.

$$f_1(0) = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \dots = 2.612 \quad \dots\dots(xiv)$$

$$\left(\frac{N}{V}\right)_{\max} = \frac{(2\pi mkT)^{3/2}}{h^3} (2.612) \quad \dots\dots(xv)$$

This shows that particle density is a function of temperature.

- In B-E distribution, it was assumed continuous distribution of energy levels in place of discrete distribution of energy levels.
- At low temperatures, the number of particles begin to accumulate into lower energy levels and a large number of particles may occupy ground state $\epsilon=0$. Therefore, it looks that energy level $\epsilon=0$ has been completely neglected in eqⁿ. (i). While this state is very important at low temperatures.

- Total number of particles for the degenerate case is given as –

$$N = N_g + N_{ex} \quad \dots\dots\dots(xvi)$$

$$\text{where } N_{ex} = \frac{4\sqrt{2}\pi Vm^{3/2}}{h^3} \int_0^\infty \frac{\varepsilon^{1/2} d\varepsilon}{e^{\alpha+\varepsilon/kT} - 1} \quad \& \quad N_g = \frac{1}{e^{-\mu/kT} - 1}$$

Therefore,

$$N_g = N - N_{ex} = N - \left[\frac{2.612V}{h^3} (2\pi mkT)^{3/2} \right] \quad \dots\dots(xvii)$$

The phenomenon in which bosons gradually accumulate in the lower energy state and a large number of these occupy the ground state ($\varepsilon=0$) is referred as '***Bose-Einstein condensation***'.

Now, condition for onset of Bose-Einstein condensation is as –

$$N > N_{ex} = \left[\frac{2.612V}{h^3} (2\pi mkT_c)^{3/2} \right] \dots\dots(xviii)$$

where

$$T_c = \frac{h^3}{2mk} \left(\frac{1}{2.612} \frac{N}{V} \right)^{2/3} \dots\dots(xix)$$

- Above T_c , number of particles in the ground state is almost negligible.
- But below T_c , particles begin to accumulate in the quantum state with $\epsilon=0$. thus, boson gas can be considered as a mixture of two phases:
 - A gaseous phase consisting of N_{ex} particles distributed over the excited states ($\epsilon>0$)
 - A condensed phase, consisting of N_g particles in the energy state with $\epsilon=0$.

- Therefore, N_{ex} and N_g can be represented in terms of total number of bosons N as –

$$\frac{N_{ex}}{N} = \left(\frac{T}{T_c} \right)^{3/2} \Rightarrow N_{ex} = N \left(\frac{T}{T_c} \right)^{3/2} \dots\dots(xx)$$

- Number of particles in the ground state N_g , is given by –

$$N_g = N - N_{ex} = N \left[1 - \left(\frac{T}{T_c} \right)^{3/2} \right] \dots\dots(xxi)$$

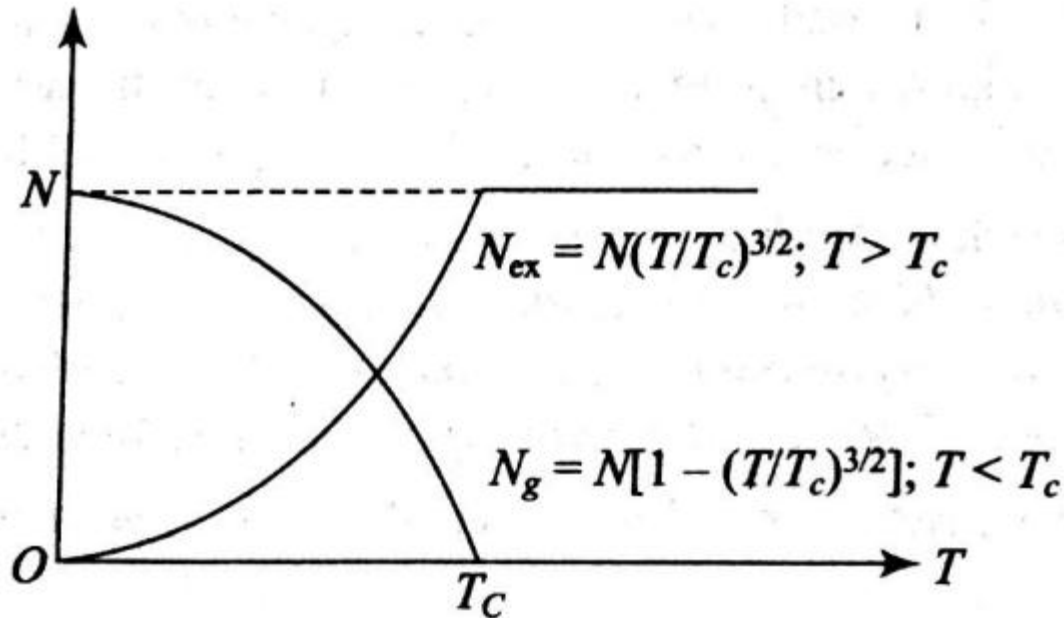


Fig.: Distribution of bosons as a function of temperature below and above the B-E condensation temperature

- i. For $T=0$, all bosons condense onto the lowest energy state ($\epsilon=0$).
- ii. As T increases, the number of particles in the ground state decreases and at $T=T_c$, their number is almost zero.
- iii. For $T>T_c$, all particles are in the excited states.

References: Further Readings

1. *Thermal Physics (Kinetic theory, Thermodynamics and Statistical Mechanics)* by S.C. Garg, R.M. Bansal and C.K. Ghosh
2. *Statistical Mechanics* by R.K. Pathria
3. *Statistical Mechanics* by K. Huang
4. *Statistical Mechanics* by Satya Prakash
5. *Elementary Statistical Mechanics* by Gupta & Kumar

Assignment

1. Obtain the partition function for a photon gas.
2. How the B-E condensation phenomenon was used to explain the peculiar behavior of liquid helium at low temperatures? Discuss in detail.

Thank You

**For any questions/doubts/suggestions and submission of
assignment
write at E-mail: neelabh@mgcub.ac.in**