Theorem (Bounded Inverse Theorem or Banach Isomorphism theorem)

Let T be one-to-one bounded linear transformation from a Banach Space X onto a Banach Space Y. Then T^{-1} is a bounded linear transformation on Y onto X.

Proof: Since T is one-one onto, the inverse function T^{-1} is a linear transformation on Y onto X. Let G be an open subset of X. Then, as T is one-one and onto, the inverse image of G under T^{-1} is $(T^{-1})^{-1}G = T(G)$. But by the open mapping theorem, it follows that $T(G)$ is open and hence T^{-1} is continuous i.e. T^{-1} is bounded.

Theorem: Let X and Y be normed linear spaces. Then $X \times Y$ is a normed linear space with the coordinate wise operations and the norm

$$\|(x,y)\| = \|x\| + \|y\|, \text{ where } x \in X \text{ and } y \in Y,$$

and $X \times Y$ is complete iff both X and Y are complete.

Proof: (i) $\|(x,y)\| = \|x\| + \|y\| \geq 0$

(ii) $\|(x,y)\| = 0 \iff \|x\| + \|y\| = 0$

$\iff \|x\| = 0$ and $\|y\| = 0$

$\iff x = 0$ and $y = 0$

$\iff (x,y) = (0,0)$

$\therefore \|(x,y)\| = 0 \iff (x,y) = 0$

(iii) $\|\alpha(x,y)\| = \|(\alpha x, \alpha y)\| = \|\alpha x\| + \|\alpha y\|$

$= |\alpha| (\|x\| + \|y\|)$

$= |\alpha| \, \|(x,y)\|$

(iv) $\|(x_1,y_1) + (x_2,y_2)\| = \|(x_1+x_2, y_1+y_2)\|$

$= \|x_1+x_2\| + \|y_1+y_2\|$

$\leq \|x_1\| + \|x_2\| + \|y_1\| + \|y_2\|$

$= \|(x_1,y_1)\| + \|(x_2,y_2)\|$

Hence $\|(x,y)\| = \|x\| + \|y\|$ is norm on $X \times Y$.

Let $\{(x_n, y_n)\}$ be a Cauchy sequence in $X \times Y$. Then for given $\epsilon > 0$, we can find a N such that

$\|(x_n, y_n) - (x_m, y_m)\| < \epsilon$ for all $m, n \geq N$.

$\Rightarrow \|(x_n - x_m, y_n - y_m)\| < \epsilon$ for all $m, n \geq N$

$\Rightarrow \|x_n - x_m\| + \|y_n - y_m\| < \epsilon$ for all $m, n \geq N$

$\Rightarrow \|x_n - x_m\| < \epsilon$ and $\|y_n - y_m\| < \epsilon$ for all $m, n \geq N$.

Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X and Y, respectively. Since X and Y are complete then $x_n \to x$ in X and $y_n \to y$ in Y in their norms, that is,

$\|x_n - x\| < \epsilon/2$ and $\|y_n - y\| < \epsilon/2$ for all $n \geq N$

Therefore,

$\|(x_n, y_n) - (x, y)\| = \|x_n - x\| + \|y_n - y\| < \epsilon$ for all $n \geq N$

This proves that $(x_n, y_n) \to (x, y) \in X \times Y$.

Therefore, $X \times Y$ is complete. The Converse follows by reversing the steps.

<u>Note</u>: The following are two other norms equivalent to the above norm

(i) $\|(x, y)\| = \max\{\|x\|, \|y\|\}$

(ii) $\|(x, y)\| = (\|x\|^p + \|y\|^p)^{1/p}, \; p > 1$.

Definition: Let X and Y be normed linear spaces and let $T : X \rightarrow Y$ be a mapping. Then the graph of T is defined to be the subset of $X \times Y$ which consists of all ordered pairs of the form (x, Tx). It is denoted by G_T. So the graph of $T : X \rightarrow Y$ is

$$G_T = \{ (x, Tx) : x \in X \}$$

Definition: Let X and Y be normed linear spaces and D a subspace of X. Then a linear transformation $T : D \rightarrow Y$ is said to be closed if and only if $x_n \in D$, $x_n \rightarrow x$ and $T(x_n) \rightarrow y$ imply $x \in D$ and $y = T(x)$.

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Theorem (The Closed graph Theorem)

If X and Y are Banach Spaces and if T is a linear transformation of X into Y, then T is continuous if and only if its graph G_T is closed.

Proof: Let G_T denote the graph of T and let T be continuous. We show that $\overline{G_T} = G_T$.

Let $(x,y) \in \overline{G_T}$, then there exists a sequence $\{(x_n, T(x_n))\}$ in G_T such that $(x_n, T(x_n)) \to (x,y)$

$\Rightarrow x_n \to x$ and $Tx_n \to y$

$\Rightarrow Tx_n \to Tx$ and $Tx_n \to y$ (Since T is continuous)

$\Rightarrow T(x) = y$

$\Rightarrow (x,y) = (x, Tx) \in G_T$

$\Rightarrow \overline{G_T} \subseteq G_T$

But $G_T \subseteq \overline{G_T}$ always, thus $\overline{G_T} = G_T$.

Therefore, G_T is closed.

Conversely, let G_T be closed subspace of $X \times Y$, then G_T is a Banach space since closed subspace of a complete space is complete.

We consider a mapping $\phi : G_T \to X$ given by

$$\phi[(x, T(x))] = x.$$

Now we shall show that ϕ is bounded linear transformation.

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$$\phi[(x, T(x)) + (y, T(y))] = \phi(x+y, T(x)+T(y))$$
$$= x+y$$
$$= \phi(x, T(x)) + \phi(y, T(y))$$

and
$$\phi[\alpha(x, Tx)] = \phi((\alpha x, \alpha T(x))) = \alpha x = \alpha\phi(x, T(x)).$$

So ϕ is linear.

And $\|\phi(x, T(x))\| = \|x\| \leq \|x\| + \|Tx\| = \|(T, Tx)\|$

Therefore, ϕ is bounded linear transformation.

Now we show that ϕ is one-one onto.

Let $\phi(x_1, Tx_1) = \phi(x_2, Tx_2)$

$\Rightarrow \phi(x_1, Tx_1) - \phi(x_2, Tx_2) = 0$

$\Rightarrow \phi[(x_1, Tx_1) - (x_2, T(x_2))] = 0$ Since ϕ is linear

$\Rightarrow \phi((x_1 - x_2, T(x_1) - T(x_2)) = 0$

$\Rightarrow \phi[(x_1 - x_2, T(x_1 - x_2))] = 0$ Since T is linear

$\Rightarrow x_1 - x_2 = 0$ by definition of ϕ

$\Rightarrow x_1 = x_2.$

So ϕ is one-one and ϕ is onto obviously.

Since ϕ is one-one and onto bounded linear transformation from a Banach space $X \times Y$ onto a Banach space X. So by Bounded inverse theorem, ϕ^{-1} is a bounded linear transformation on X into $X \times Y$. Therefore ϕ^{-1} is continuous.

Now let $x_n \to x$

$\Rightarrow \phi^{-1}(x_n) \to \phi^{-1}(x)$ since ϕ^{-1} is continuous

$\Rightarrow (x_n, T(x_n)) \to (x, T(x))$

$\Rightarrow T(x_n) \to T(x)$.

Therefore, T is continuous.

This completes the proof of the theorem.