Product Topology continued …

DR. SHEO KUMAR SINGH
Assistant Professor,
Department of Mathematics
School of Physical Sciences,
Mahatma Gandhi Central University, Motihari, Bihar-845401.

Email: sheokumarsingh@mgcub.ac.in

Product Topology

In my previous PPT, I have defined the product topology on the cartesian product $X \times Y$, for topological spaces X and Y in two ways which are given as follows:

- the collection $\mathfrak{B} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ is a base for the product topology on $X \times Y$.
- the collection $\mathcal{S} = \{{p_1}^{-1}(U) \mid U \text{ is open in } X\} \cup \{{p_2}^{-1}(V) \mid V \text{ is open in } Y\}$ is a subbase for the product topology on $X \times Y$, where $p_1: X \times Y \to X$ and $p_2: X \times Y \to Y$ defined respectively, by $p_1(x, y) = x$ and $p_2(x, y) = y$ are the projection maps in the components X and Y.

Note: The basis element $U \times V = {p_1}^{-1}(U) \cap {p_2}^{-1}(V)$.

Here, I will generalize the above definition to a more general setting.

Consider the cartesian products $X_1 \times \cdots \times X_n$ and $X_1 \times X_2 \times \cdots$, where first is a finite product, and second is an arbitrary product of topological spaces $X_1, X_2, \ldots$.

We can proceed in two ways:

- One way is to take all the sets of the form $U_1 \times \cdots \times U_n$ in the first case, and of the form $U_1 \times U_2 \times \cdots$ in the second case as basis elements, where U_i is open in X_i.

- Another way is to generalize the subbase formulation, where we take all the sets of the form ${p_i}^{-1}(U_i)$, where U_i is open in X_i as subbasis elements.

Here, we can observe that:

- In case of finite products, a typical basis element (i.e., basic open set) coincides in both the approaches and is equal to $U_1 \times \cdots \times U_n$.

- In case of an arbitrary product, a typical basis element is $U_1 \times U_2 \times \cdots$ using first approach while using the second approach, a typical basis element is a finite intersection of subbasis elements ${p_i}^{-1}(U_i)$, say for $i = i_1, \ldots, i_r$.

Here note that ${p_i}^{-1}(U_i) = X_1 \times \cdots \times X_{i-1} \times U_i \times X_{i+1} \times \cdots$ for all i.
Thus, using the second approach, a typical basis element comes out to be $U_1 \times U_2 \times \cdots$, where $U_i = X_i$ if $i \neq i_1, \ldots, i_r$, i,e., $U_i = X_i$ except for finitely many values of i.

- Topology obtained using the first approach is said to be the *box topology* on the cartesian products $X_1 \times \cdots \times X_n$ and $X_1 \times X_2 \times \cdots$.

- Topology obtained using the second approach is said to be the *product topology* on the cartesian products $X_1 \times \cdots \times X_n$ and $X_1 \times X_2 \times \cdots$.

Thus, it is clear from the above discussion that the two topologies (i.e., product topology and box topology) are precisely the same for a finite product. But they differ for an arbitrary product and in this case the box topology is finer than the product topology.

Now the question is:

Why we prefer the product topology in comparison to the box topology?

The answer to the above question is:

There are several important theorems about finite products which will also hold for arbitrary products if we use the product topology, but not if we use the box topology.

For example, if X and $X_{i's}$ are topological spaces, then a map $f: X \to \prod_{i\in I} X_i$ is continuous if and only if $p_i \circ f$ is continuous for all $i \in I$, where p_i is the i^{th} projection map.

References:

- James R. Munkres, *Topology*, 2nd ed., PHI.

THANK YOU