Homogeneous Fredholm Integral Equations of the Second kinds with Degenerate Kernels

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Q1: Find the eigenvalues and eigenfunctions of the homogeneous integral equation $y(x) = \lambda \int_0^1 e^x e^t y(t)dt$

Sol: Given that $y(x) = \lambda e^x \int_0^1 e^t y(t)dt$ (1)

Let $C = \int_0^1 e^t y(t)dt$ (2)

Then equation (1) reduces to $y(x) = C\lambda e^x$ (3)

From equation (3) , $y(t) = C\lambda e^t$ (4)

By equation(4), then equation (2) becomes $C = \int_0^1 e^t (C\lambda e^t)dt$

$$C\left[1-\frac{\lambda}{2}(e^2-1)\right]=0 \qquad (5)$$

If $C=0$ then equation (4) gives $y(x)=0$. There , assume that for non-zero solution of equation (1), $C\neq 0$. Hence equation (5) reduces to

$$\lambda=\frac{2}{(e^2-1)} \qquad (6)$$

Which is an eigenvalue of equation (1)

Putting the value of λ given by equation (6) in (3), the corresponding eigenfunction is given by

$$y(x)=\frac{2C}{(e^2-1)}e^x$$

Therefore, corresponding to eigenvalue $\lambda=\frac{2}{(e^2-1)}$ there corresponds the eigenfunction e^x.

Q2: Solve the homogeneous Fredholm integral equation of the second kind

$$y(x) = \lambda \int_0^{2\pi} \sin(x+t)\, y(t)dt \qquad (1)$$

Sol: Given that equation (1), then

$$y(x) = \lambda \int_0^{2\pi} [\sin x \cos t + \cos x \sin t]\; y(t)dt$$

$$y(x) = \lambda \sin x \int_0^{2\pi} \cos t\, y(t)dt + \lambda \cos x \int_0^{2\pi} \sin t\, y(t)dt \qquad (2)$$

Let $C_1 = \int_0^{2\pi} \cos t\, y(t)dt$ and $C_2 = \int_0^{2\pi} \sin t\, y(t)dt$ (3a,3b)

Therefore, then equation (2) reduces to

$$y(x) = \lambda C_1 \sin x + \lambda C_2 \cos x \qquad (4)$$

$$y(t) = \lambda C_1 \sin t + \lambda C_2 \cos t \qquad (5)$$

Using equation (5), then equation (3a) becomes

$$C_1 = \int_0^{2\pi} \cos t\,(\lambda C_1 \sin t + \lambda C_2 \cos t)dt$$

$$C_1 - \lambda C_2 \pi = 0 \qquad (6)$$

Using equation (5), then equation (3b) becomes

$$C_2 = \int_0^{2\pi} \sin t\,(\lambda C_1 \sin t + \lambda C_2 \cos t)dt$$

$$\lambda C_1 \pi - C_2 = 0 \qquad (7)$$

Therefore, we have a system of homogeneous linear equations (6) and (7) for determining C_1 and C_2. From non-zero solution of the system of equations, $\begin{vmatrix} 1 & -\lambda\pi \\ \lambda\pi & -1 \end{vmatrix} = 0, \quad \lambda = \pm\frac{1}{\pi}$

The eigenvalue are given by $\lambda_1 = \frac{1}{\pi}$ and $\lambda_2 = -\frac{1}{\pi}$ (8)

To determine the eigenfunction corresponding to $\lambda = \lambda_1 = \frac{1}{\pi}$, in equation (6) and (7), we obtain

$$C_1 - C_2 = 0 \quad (9)$$

and

$$C_1 - C_2 = 0 \quad (10)$$

Both equation (9) and (10) gives $C_2 = C_1$, from equation (4), we obtain

$$y(x) = \frac{C_1}{\pi}(\sin x + \cos x)$$

Taking $\frac{C_1}{\pi} = 1$, the required eigenfunction $y_1(x)$ is given by

$$y_1(x) = (\sin x + \cos x) \quad (11)$$

To determine the eigenfunction corresponding to $\lambda = \lambda_2 = -\frac{1}{\pi}$,

in equation (6) and (7), we get $\quad C_1 + C_2 = 0 \qquad (12)$

and $\quad C_1 + C_2 = 0 \qquad (13)$

Both equation (12) and (13) gives $C_2 = -C_1$, from equation (4),

we obtain

$$y(x) = \frac{C_1}{\pi}(\sin x - \cos x)$$

Taking $\frac{-C_1}{\pi}$=1, the required eigenfunction $y_2(x)$ is given by

$$y_2(x) = (\sin x - \cos x) \qquad (14)$$

From equation (8), (11) and (14), the required eigenvalues and

eigenfunctions are given $\quad \lambda_1 = \frac{1}{\pi} \quad y_1(x) = (\sin x + \cos x)$

and

$$\lambda_2 = -\frac{1}{\pi} \quad y_2(x) = (\sin x - \cos x)$$

Q3: Prove that the homogeneous integral equation $y(x) = \lambda \int_0^1 \left(t\sqrt{x} - x\sqrt{t}\right) y(t)dt$ does not have real eigenvalues and eigenfunctions.

Sol: Given that $y(x) = \lambda \int_0^1 \left(t\sqrt{x} - x\sqrt{t}\right) y(t)dt$

$$y(x) = \lambda\sqrt{x} \int_0^1 t y(t)dt - \lambda x \int_0^1 \sqrt{t} y(t)dt \qquad (1)$$

Let $C_1 = \int_0^1 t y(t)dt$ and $C_2 = \int_0^1 \sqrt{t} y(t)dt$ (2, 3)

Then equation (1) reduces to $y(x) = \lambda C_1 \sqrt{x} - \lambda C_2 x$ (4)

From equation (4) $y(t) = \lambda C_1 \sqrt{t} - \lambda C_2 t$ (5)

Using equation (5), equation(2) becomes

$$C_1 = \int_0^1 t\left(\lambda C_1 \sqrt{t} - \lambda C_2 t\right)dt$$

$$\left(1-\frac{2\lambda}{5}\right)C_1+\frac{\lambda}{3}C_2=0 \qquad (6)$$

Using equation (5), equation(3) becomes

$$C_2=\int_0^1 \sqrt{t}\left(\lambda C_1\sqrt{t}-\lambda C_2 t\right)dt$$

$$-\frac{\lambda}{2}C_1+\left(1+\frac{2\lambda}{5}\right)C_2=0 \qquad (7)$$

For non-zero solution of the system of equation (6) and (7) Using equation (5), equation(3) becomes

$$D(\lambda)=\begin{vmatrix}1-\frac{2\lambda}{5} & \frac{\lambda}{3}\\ -\frac{\lambda}{2} & 1+\frac{2\lambda}{5}\end{vmatrix}=0, \qquad \lambda=\pm i\sqrt{150}$$

Showing that $D(\lambda)\neq 0$ for any real value of λ . Therefore the system of equations (6) and (7) has unique solution $C_1 = C_2 = 0$ for all real λ. Hence, from equation (4) $y(x)=0$, which is zero solution. Therefore, the given equation does not have real eigenvalue and eigenfunctions

Q1: Determine the eigenvalues and eigenfunctions of the homogeneous integral equation

(A)

$$y(x) = \lambda \int_0^1 K(x,t)dt \quad \text{where } K(x,t) = \begin{cases} t(x+1), & 0 \le x \le t \\ x(1+t), & t \le x \le 1 \end{cases}$$

(B)

$$y(x) = \lambda \int_0^1 K(x,t)dt \quad \text{where } K(x,t) = \begin{cases} -e^{-t}\sinh x\,, & 0 \le x \le t \\ -e^{-x}\sinh t\,, & t \le x \le 1 \end{cases}$$

Q2: Show that the integral equation

$$y(x) = \lambda \int_0^{2\pi} \sin x \sin 2t \, y(t)dt$$

has no eigenvalues. (Try to yourself)

Thank you