Compact Spaces

DR. SHEO KUMAR SINGH
Assistant Professor,
Department of Mathematics
School of Physical Sciences,
Mahatma Gandhi Central University, Motihari, Bihar-845401.

Email: sheokumarsingh@mgcub.ac.in

Let X be a topological space and $\mathcal{A} = \{G_i\}_{i\in I}$ be a family of subsets of X, then

- $\{G_i\}_{i\in I}$ is said to be a cover of X, if $\cup_{i\in I} G_i = X$.
- a sub-family $\{G_j\}_{j\in J\subseteq I}$ of cover $\mathcal{A} = \{G_i\}_{i\in I}$ is said to be a subcover if $\cup_{j\in J} G_j = X$.

Note: If all the members of a cover are open subsets of X, then it is said to be an open cover of X.

A topological space X is said to be *compact* if every open cover of X has a finite subcover.

Example: Every finite topological space is compact.

Result: $\mathbb{R}$ (with usual topology) is not compact.

Solution: The open cover $\{(a,b) \mid a,b \in \mathbb{Q}\}$ of $\mathbb{R}$ has no finite subcover.

A family $\mathcal{F} = \{F_i\}_{i\in I}$ of subsets of a set X is said to has a finite intersection property (*f.i.p.*, in short) if $\mathcal{F} \neq \emptyset$ and $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_r} \neq \emptyset$ for every finite subset $\{F_{i_1}, F_{i_2}, \dots, F_{i_r}\}$ of $\mathcal{F}$.

Theorem 1: A topological space X is compact if and only if every family of closed subsets of X which has the f.i.p. has non-empty intersection.

Proof: (Outlines only)

Let $\mathcal{F}$ be a family of closed subsets of X having f.i.p. Suppose that $\cap_{F\in\mathcal{F}} F = \emptyset$, then $\cup_{F\in\mathcal{F}} (X - F) = X$, i.e., $\{X - F \mid F \in \mathcal{F}\}$ becomes an open cover of X. As X is compact, so, there are finitely many $F_1, \dots, F_k \in \mathcal{F}$ such that $(X - F_1) \cup \cdots \cup (X - F_k) = X$. But then $F_1 \cap \cdots \cap F_k \neq \emptyset$, which is not possible as $\mathcal{F}$ has f.i.p. Converse part is left as an easy exercise. ■

Some important results

Result 1: Continuous image of a compact space is compact.

Result 2: Closed subspace of a compact space is compact.

Result 3: Compact subspace of a Hausdorff space is closed subspace.

Proof: (Outlines only)

(1): Let $f: X \to Y$ be cont. onto map and X be compact. Let $\mathcal{A}$ be an open cover of Y. Then clearly, $\bar{\mathcal{A}} = \{f^{-1}(A) \mid A \in \mathcal{A}\}$ is an open cover of X. So, there are finitely many $A_1, \dots, A_k \in \mathcal{A}$ such that $X = f^{-1}(A_1) \cup \cdots \cup f^{-1}(A_k)$. But then $Y = f(X) = f(f^{-1}(A_1)) \cup \cdots \cup f(f^{-1}(A_k)) = A_1 \cup \ldots \cup A_k$. Hence, Y is compact.

(2): Let Y be a closed subspace of a compact space X. Let $\mathcal{A} = \{V_i\}_{i \in I}$ be an open cover of Y. Then for each $i \in I$, there are open subsets G_i of X such that $V_i = G_i \cap Y$. Since, Y is closed in X, so, $\{G_i \mid i \in I\} \cup (X - Y)$ forms an open cover of X. So, there are finitely many $i_1, \dots, i_k \in I$ such that $X = G_{i_1} \cup \cdots \cup G_{i_k} \cup (X - Y)$. But then, $Y \subseteq G_{i_1} \cup \cdots \cup G_{i_k}$ and consequently, $Y = V_{i_1} \cup \cdots \cup V_{i_k}$. Thus, Y is compact.

(3): (Left as an exercise).

Some important results

Lemma: If A is a compact subset of a space X and y be a pint of a space Y, then for each open set W in $X \times Y$ containing $A \times \{y\}$ there are open sets U in X and V in Y such that $A \times \{y\} \subseteq U \times V \subseteq W$.

Theorem 2: If space X is compact, then the projection $p_Y: X \times Y \to Y$ is closed for every topological space Y.

Proof: (Outlines only)

Let X be compact and Y be an arbitrary space and F be closed in $X \times Y$. Suppose $y \notin p(F)$, then as $X \times \{y\} \subseteq X \times Y - F$, it follows from the above lemma that there is some open set V of Y such that $X \times V \cap F = \emptyset$. So, $p(F) \cap V = \emptyset$. Hence, y cannot be a limit point of $p(F)$. Thus, $p(F)$ is closed in Y.

References:

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THANK YOU