Connected Spaces

DR. SHEO KUMAR SINGH
Assistant Professor,
Department of Mathematics
School of Physical Sciences,
Mahatma Gandhi Central University, Motihari, Bihar-845401.

Email: sheokumarsingh@mgcub.ac.in

Let X be a topological space. Then $A, B \subseteq X$ are said to be separated if $\bar{A} \cap B = \emptyset = A \cap \bar{B}$. In such a case, the pair (A, B) is called a separation of X.

Definitions:

- A topological space X is said to be connected if it cannot be expressed as a union of disjoint non-empty open sets.
- A topological space X is said to be disconnected if it is not connected.

Theorem 1: X is disconnected $\Leftrightarrow$ X has a separation.

Proof: (Leave as an exercise to the reader).

Some important results

Theorem 2: For a topological space X the following conditions are equivalent:

(i) The space X is connected.

(ii) Only closed-and-open (i.e., clopen) subsets of X are $\emptyset$ and X.

(iii) If $X = A \cup B$ and A and B are separated sets, then one of them is empty.

(iv) Every cont. map $f: X \to 2_D$ is constant, where 2_D is two-point discrete space.

Proof: (Outlines only)

(i) $\Rightarrow$ (ii): If X has a clopen subset G, then the pair $(G, X - G)$ turns out to be a separation of X, which is a contradiction.

(ii) $\Rightarrow$ (iii): Now, A and B are separated sets, so, $\bar{A} \cap B = \emptyset = A \cap \bar{B}$. Also, as $X = A \cup B$, so A and B turn out to be clopen sets in X. Hence, by (ii) either A or B is empty.

(iii) ⇒ (iv): Suppose that $2=\{a,b\}$. If $f: X \to 2_D$ is a non-constant cont. map, then $f^{-1}\{a\}$ & $f^{-1}\{b\}$ are non-empty separated sets and $X = f^{-1}\{a\} \cup f^{-1}\{b\}$, contradiction.

(iv) ⇒ (i): If X is disconnected, then $X = U \cup V$, for some disjoint non-empty open subsets U and V of X. But then, the non-constant map $f: X \to 2_D$ sending U to $\{a\}$ and V to $\{b\}$ turns out to be continuous, which contradicts (iv).

Theorem 3:

(i) Connectedness is preserved under continuous maps.

(ii) If D is disconnected subset of a space X with separation U & V and C is connected with $C \subseteq D$, then either $C \subseteq U$ or $C \subseteq V$.

(iii) If A is a connected subset of a space X and $A \subseteq B \subseteq \bar{A}$, then B is also connected. In particular, $\bar{A}$ is also connecetd.

(iv) If $\{A_i\}_{i \in I}$ is a collection of connected subsets of a topological space X such that $\cap_{i \in I} A_i \neq \emptyset$, then $\cup_{i \in I} A_i$ is connected in X.

(v) If X and Y are connected spaces, then the product space $X \times Y$ is also connected.

Proof: (Outlines only)

(i) Let $f: X \to Y$ be a cont. onto map and X be connected. Suppose that Y is not connected and (U,V) is a separation of Y. Then it can be verified that $f^{-1}(U)$ & $f^{-1}(V)$ forms a separation of X, contradiction. So, Y is connected.

(ii) (Left as an exercise to the students).

(iii) Suppose that (C, D) is a separation of B in X, i.e., $B = C \cup D$. Then $A \subseteq C$ or $A \subseteq D$ (using (ii)). So, let $A \subseteq C$. Then $A \cap D = \emptyset$. Here $B \cap C \neq \emptyset \neq B \cap D$. So, pick any $x \in B \cap D$, then $x \in B$ and $x \in D$. As $A \cap D = \emptyset$, so $x \notin \bar{A}$, but $x \in B \subseteq \bar{A}$, so $x \in \bar{A}$, contradiction. Hence, B is connected.

(iv) If $\cup_{i \in I} A_i$ is not connected, then we must have a separation (U, V) of $\cup_{i \in I} A_i$ in X. As $\cap_{i \in I} A_i \neq \emptyset$, so pick any $x \in \cap_{i \in I} A_i$. Then either $x \in U$ or $x \in V$. Suppose that $x \in U$. As $x \in A_i$ and is connected, so $A_i \subseteq U, \forall i \in I$, so $\cup_{i \in I} A_i \subseteq U$, but then (U, V) cannot be a separation of $\cup_{i \in I} A_i$, contradiction. Thus, $\cup_{i \in I} A_i$ is connected.

(v) We know that $\{x\} \times Y \cong Y$ and $X \times \{y\} \cong X, \forall x \in X \; \& \; \forall y \in Y$, so the sets $\{x\} \times Y$ and $X \times \{y\}$ and hence $(\{x\} \times Y) \cup (X \times \{y\})$ are connected in $X \times Y$. Now the connectedness of $X \times Y$ follows from the fact that $X \times Y = \cup_{y \in Y} [(\{x\} \times Y) \cup (X \times \{y\})]$, for any $x \in X$.

Note: From (v) we conclude that, if $X_1, \ldots, X_n$ are connected spaces, then the product space $X_1 \times \cdots \times X_n$ is also connected.

References:

- James R. Munkres, *Topology*, 2nd ed., PHI.
- Colin Adams and Robert Franzosa, *Introduction to Topology: Pure and Applied*, Pearson.
- Ryszard Engelking, *General Topology*, Heldermann Verlag.

THANK YOU