COMPLEX ANALYSIS

continued…

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Defⁿ Mean-value property. A complex function $f(z)$ defined in a domain D is said to have mean value property, if for every $a \in D$ and $r > 0$ s.t. $N_r(a) = \{z : |z-a| \le r\} \subseteq D$, and

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\, d\theta$$

$f(a)$ is the mean value of $f(z)$ on the circle $N_r(a)$

Gauss mean value theorem

If $f(z)$ is an analytic function in a domain D, then $f(z)$ has mean value property.

Proof. Let $a \in D$ {arbitrary point} s.t. $\{z : |z-a| = r,\ r > 0\} \subseteq D$

$z - a = re^{i\theta} \Rightarrow z = a + re^{i\theta}$

using Cauchy's Integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\, dz = \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(a+re^{i\theta})}{\cancel{re^{i\theta}}} \cdot \cancel{ire^{i\theta}}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\, d\theta$$

Maximum Modulus Principle

In real analysis of one variable we simply talk about max or min.
But in complex analysis (not ordered) it is difficult but we can think max or min of mod of $f(z)$ real part or imaginary part of $f(z)$.

Def^n: Let D be a subset of $\mathbb{C}$. A complex fns defined on D is said to have a local maximum modulus at $a \in D$ if $\exists\ \delta > 0$ s.t. $N_\delta(a) \subset D$ and $|f(z)| \le |f(a)|\ \forall\ z \in N_\delta(a)$
a local min similarly defined.

Theorem (Maximum Modulus Principle)

Let f is analytic in a domain D and a is a point in D s.t. $|f(z)| \le |f(a)|$ holds $\forall\, z \in D$. Then, f is a const.

Lemma: Let $g(x)$ be a real contin fns defined on $[a,b] \ni g(x) \geq 0 \;\forall\, x \in [a,b]$. If $\int_a^b g(t)\,dt = 0$, then $g(x) = 0 \;\forall\, x \in [a,b]$

Proof of th: $\because a \in D$ and D is open, $\exists\, r > 0$ s.t. $N_r(a) \subset D$. Then f is analytic inside $C = \{z : |z-a| = r\}$ and on the circle C. So by Cauchy's Integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-a}\,ds = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

By hypothesis $|f(a+re^{i\theta})| \le |f(a)|$

$$|f(a)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{i\theta})|\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f(a)|\,d\theta$$

$$= |f(a)|$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi}\left[|f(a+re^{i\theta})| - |f(a)|\right] d\theta = 0 \qquad \left\{\begin{array}{l}\text{as the L.H.S.}\\ \text{cannot be}\\ \text{(−)ve}\end{array}\right\}$$

∵ Integrand is contin and non (−)ve

using lemma

$$|f(a+re^{i\theta})| = |f(a)| \quad \forall\ 0 \le \theta \le 2\pi$$

as the circle chosen is arbitrary

$\Rightarrow$ f is const. on the whole of D.

Note: Min value of $|f|$ may be attained at an interior pt. of D without f being const.

for ex. let $f(z) = z$ for $z \in \Delta_r$. Then

$|f(z)| = |z| \geq 0 = |f(0)| \Rightarrow \min |f(z)|$ attains at origin

The max of $|f(x+iy)| = \sqrt{x^2+y^2}$ is attained at the boundary pt. $|z| = r$.

Maximum modulus Theorem Let f is analytic in a bounded domain D and contin on $\overline{D}$. then, $|f(z)|$ attains its maximum at some pt. on the boundary ∂D of D.

contd

Proof. We know that a contin fn on a compact (closed + bounded) set attains a maximum.
As given f is bounded on $\overline{D}$ $\Rightarrow$ max of f attained at some pt of $\overline{D}$.
By Max. Mod. Principle it cannot be in D
$\Rightarrow$ Max value attains at the boundary of of D i.e ∂D.

Ex. Let $f(z) = z^2$ defined on $D = \{z : |z-1-i| \le 1\}$
Let us discuss the max value of $|f(z)|$.

let $z = 1+i+e^{i\theta} = 1+i+\cos\theta + i\sin\theta$
$= (1+\cos\theta) + i(1+\sin\theta)$ $\quad \theta \in [0, 2\pi)$

Then $|f(z)| = \sqrt{[(1+\cos\theta)^2 - (1+\sin\theta)^2 + 2i(1+\cos\theta)(1+\sin\theta)] \cdot [(1+\cos\theta)^2 - (1+\sin\theta)^2 - 2i(1+\cos\theta)(1+\sin\theta)]}$
$= \sqrt{f(z)\cdot\overline{f(z)}}$

$= \sqrt{[(1+\cos\theta)^2 - (1+\sin\theta)^2]^2 + 4(1+\cos\theta)^2(1+\sin\theta)^2}$

$= \sqrt{[(1+\cos\theta)^2 + (1+\sin\theta)^2]^2} = (1+\cos\theta)^2 + (1+\sin\theta)^2$
$= 3 + 2(\sin\theta + \cos\theta)$

$\Rightarrow$ max of $|f(z)|$ attains at $\theta = \pi/4$, value is $3+2\sqrt{2}$ and the pt. is $1+i+e^{i\pi/4}$ [boundary]

Minimum modulus principle Let $f(z)$ be analytic in D s.t. $f(z) \neq 0 \ \forall \ z \in D$. Then $|f(z)|$ does not attain its min. unless it is const.

Proof. $\because f(z) \neq 0 \ \forall \ z \in D$, $\Rightarrow \frac{1}{f(z)}$ is analytic in D.

By Maximum mod principle $\left|\frac{1}{f(z)}\right|$ does not attain its max unless it is a const.

$\Rightarrow |f(z)|$ doesn't attain its min unless it is a const.

Th// Minimum Modulus Theorem - Let $f(z)$ be an analytic fn in D and contin on ∂D s.t. $f(z) \neq 0 \ \forall \ z \in D$. Then $|f(z)|$ attains its min value on ∂D.

Proof. If $f(z) = 0$ for some $z \in \partial D$
then $\min|f(z)| = 0$. {attains at the boundary}
In case if $f(z) \neq 0$ for any $z \in \partial D$
then as given $f(z) \neq 0 \ \forall \ z \in D \cup \partial D$.
Now let $g(z) = 1/f(z)$ {contin on ∂D and analytic in D}
Hence by Max. mod. th. max $g(z)$ occurs somewhere on ∂D
$\Rightarrow$ min of $f(z)$ occurs somewhere on ∂D.

Ex: let $f(z)$ be a nonconst. analytic fn defined in a domain $D = \{z : |z| < r\}$ and contin on ∂D s.t. $|f(z)| > m$ on ∂D. If $|f(0)| < m$, show that there exist atleast one zero of $f(z)$ in $|z| < r$. [Hence $\exists$ atleast one zero in $|z| < r$]

Sol: let $f(z) \neq 0$ in $|z| < r$. to $\because$ $f(z)$ is analytic in D and contin on ∂D by min mod th. min of $|f(z)|$ occure on $|z| = r$. contradicts that $|f(0)| < m$ and $|f(z)| > m$ on $|z| = r$

Corrollary: Let f is analytic in a bounded domain D and contin. on $\bar{D}$. Then each of $\operatorname{Re} f(z)$, $-\operatorname{Re} f(z)$, $\operatorname{Im} f(z)$ and $-\operatorname{Im} f(z)$ attains its max at some pt. on the boundary of D.

Proof. Let $u(x,y) = \operatorname{Re} f(z)$ and let $g(z) = e^{f(z)}$

By Maximum mod ~~principle~~ Theorem,

$|g(z)| = |e^{u(x,y)} \cdot e^{iv(x,y)}| = e^{u(x,y)}$ cannot assume max value in D.

$\because e^{u}$ is max when u is max

$\Rightarrow u(x,y)$ cannot assume its max value in D.

other can be done similarly.

Note:- Harmonic fns $u(x,y)$ cannot attain its max or min in D. i.e. either the fns is const. or attains max or min on the boundary.

THANK YOU !