COMPLEX ANALYSIS

continued…

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Taylor and Laurent Series

Expansion of analytic functions as power series

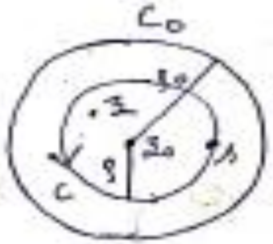
power series: A series of the type $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

Th-1 (Taylor's Theorem) Let $f(z)$ be analytic at all points within a circle C_0 with centre z_0 and radius ρ_0. Then for every point z within C_0, we have
$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots$$

C_0

Proof: let $z \in \text{int}\, C_0$.

let $|z-z_0| = r$, and let C be the circle with centre z_0 and radius ρ, s.t. $r < \rho < \rho_0$ (so that z lies inside C)



By Cauchy's Integral formula,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}\, ds \quad \text{—— (1)}$$

now we can write

$$\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)} = \frac{1}{s-z_0}\left[\frac{1}{1-\frac{z-z_0}{s-z_0}}\right]$$

$$= \frac{1}{s-z_0}\left[1+\frac{z-z_0}{s-z_0}+\frac{(z-z_0)^2}{(s-z_0)^2}+\cdots\frac{(z-z_0)^{n-1}}{(s-z_0)^{n-1}} + \frac{(z-z_0)^n}{(s-z_0)^n}\cdot\frac{1}{1-\frac{z-z_0}{s-z_0}}\right]$$

$$\left\{\begin{aligned}(1-x)^{-1} &= 1+x+x^2+x^3+\cdots x^{n-1}+x^n+x^{n+1}+x^{n+2}+\cdots\\ &= 1+x+x^2+\cdots x^{n-1}+x^n(1+x+x^2+\cdots x^{n-1}+x^n+\cdots)\\ &= 1+x+x^2+\cdots+x^{n-1}+x^n\cdot(1-x)^{-1}\\ &= 1+x+x^2+\cdots+x^{n-1}+x^n\cdot\frac{1}{1-x}\end{aligned}\right. \quad \left\}\begin{aligned}x &= \frac{z-z_0}{s-z_0}\\ |x| &< 1\end{aligned}\right.$$

$$= \frac{1}{s-z_0} + \frac{z-z_0}{(s-z_0)^2} + \frac{(z-z_0)^2}{(s-z_0)^3} + \cdots \frac{(z-z_0)^{n-1}}{(s-z_0)^n} + \frac{(z-z_0)^n}{(s-z_0)^{n+1}} \cdot \frac{s-z_0}{s-z} \quad \text{--- (2)}$$

multiplying each term in (2) by $\frac{f(s)}{2\pi i}$ and integrating term by term around c. and using (1) we get

$$\frac{1}{2\pi i}\int_c \frac{f(s)\,ds}{s-z} = \frac{1}{2\pi i}\int_c \frac{f(s)}{s-z_0}\,ds + \frac{(z-z_0)}{2\pi i}\int_c \frac{f(s)}{(s-z_0)^2}\,ds$$

$$\parallel$$

$$f(z) \qquad + \frac{(z-z_0)^2}{2\pi i}\int_c \frac{f(s)}{(s-z_0)^3}\,ds + \cdots + \frac{(z-z_0)^{n-1}}{2\pi i}\int_c \frac{f(s)\,ds}{(s-z_0)^n}$$

$$+ \frac{(z-z_0)^n}{2\pi i}\int_c \frac{f(s)\,ds}{(s-z_0)^n \cdot (s-z)}$$

using derivatives for analytic fns, we have

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + (z-z_0)^2 \cdot \frac{f''(z_0)}{2!} + \cdots$$
$$+ (z-z_0)^{n-1} \cdot \frac{f^{(n-1)}(z_0)}{(n-1)!} + R_n \quad \Big\} \text{ --- } (3)$$

where $R_n = \frac{(z-z_0)^n}{2\pi i} \int_C \frac{f(s)\,ds}{(s-z)(s-z_0)^n} \Big\} \text{ --- } (4)$

now we have to show $R_n \to 0$ as $n \to \infty$

we have $|z-z_0| = r$, $|s-z_0| = \rho$

$\therefore |s-z| = |(s-z_0)-(z-z_0)| \geq |s-z_0| - |z-z_0|$
$= \rho - r$.

let M denotes max value of $f(s)$ on C,

from (4) we get

$$|R_n| = \left| \frac{(z-z_0)^n}{2\pi i} \int_C \frac{f(s)\,ds}{(s-z)(s-z_0)^n} \right|$$

$$\le \frac{|z-z_0|^n}{|2\pi i|} \int_c \frac{|f(s)|\,|ds|}{|s-z|\,|s-z_0|^n}$$

$$\le \frac{r^n}{2\pi} \int_c \frac{M\,|ds|}{(\rho-r)\rho^n}$$

{In D^r $|s-z|$ replaced by its max value so the inequality remains}

$$= \frac{r^n}{2\pi} \frac{M}{(\rho-r)\rho^n} \left\{ \int_c |ds| = 2\pi\rho \right\}$$

$$= \frac{r^n}{\cancel{2\pi}} \frac{M}{(\rho-r)\rho^n} \cdot \cancel{2\pi}\rho = \frac{M\rho}{\rho-r}\left(\frac{r}{\rho}\right)^n$$

$\because r < \rho \Rightarrow$ R.H.S. $\to 0$ as $n \to \infty$

$\Rightarrow R_n \to 0$ as $n \to \infty$.

Thus as $n \to \infty$, the limit of the sum of the first n terms on the R.H.S. of (3) is $f(z)$.

$$\boxed{\therefore\ f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0)}$$

known as Taylor series. (F)

Remark ①: when $z_0 = 0$ (*) reduces to

$$\boxed{f(z) = f(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} f^n(0)} \quad \text{Maclaurin's series.}$$

Remark-2 To write $f(z)$ as an expansion as a Taylor's series, it is essential for a fun to be analytic at all pts inside the circle C_0 then the convergence of Taylor's series assured.

Hence the greatest radius of convergence of series is the distance from the pt. z_0 to the nearest pt where the fun is not analytic.

Example Expand $\log(1+z)$ in a Taylor series about $z=0$, also find the region of convergence for the series.

Sol^n. let $f(z) = \log(1+z)$

$f'(z) = \frac{1}{1+z}$

$f''(z) = -\frac{1}{(1+z)^2}$

$f^{(n)}(z) = (-1)^{n-1}\frac{(n-1)!}{(1+z)^n}$

Then $f(0) = 0$, $f'(0) = 1$

$f''(0) = -1$, $f'''(0) = 2!$

-- $f^{(n)}(0) = (-1)^{n-1}(n-1)!$

Therefore. $f(z) = \log(1+z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0)$

$+ \cdots \frac{z^n}{n!} f^{(n)}(0) + \cdots$

$= 0 + z + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}.2! + \cdots \frac{z^n}{n!}(-1)^{n-1}(n-1)! + \cdots$

$= z - \frac{z^2}{2!} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots + (-1)^{n-1}\frac{z^n}{n} + \cdots$

For convergence of series.

{ Refer convergence of infinite series for complex values fns }

let $u_n = \dfrac{(-1)^{n-1} z^n}{n}$, $u_{n+1} = \dfrac{(-1)^n z^{n+1}}{n+1}$

$$\therefore \lim_{n\to\infty} \left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n z}\right| = \lim_{n\to\infty} \left|\frac{1+\frac{1}{n}}{z}\right| = \frac{1}{|z|}$$

$\Rightarrow$ by using ratio test, the series converges for $|z| < 1$.

for $|z| = 1$, for $z = -1$ singularity of $\log(1+z)$ nearest the pt $z = 0$.

$\Rightarrow$ the ~~series~~ series converges for all values of z within the circle $|z| = 1$

Ex. write following in Taylor series expansion about $z = 0$, also give the region of convergence

(i) e^z (ii) $\sin z$ (iii) $\cos z$ {Do by yourself}

Ex. Consider $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{1}{1+x^4}$ {Real power series}

power series about $a=0$ is given by

$$f(x) = (1+x^4)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

{Interval of convergence $(-1,1)$ as $|x| < 1$}

On the other hand, its complex analog

$f(z) = \frac{1}{1+z^4}$ ~~has~~ is not analytic at

$$z_k = e^{i\pi(2k+1)/4} \quad k = 0,1,2,3.$$

$$\boxed{\begin{aligned} z^4 &= -1 \\ z_k^4 &= e^{i(2k+1)\pi} \\ z_k &= e^{i(2k+1)\pi/4} \end{aligned}}$$

Clearly, the distance from 0 to the nearest singularity is 1. which is the radius of convergence for the correspon series about 0.

Laurent's Theorem

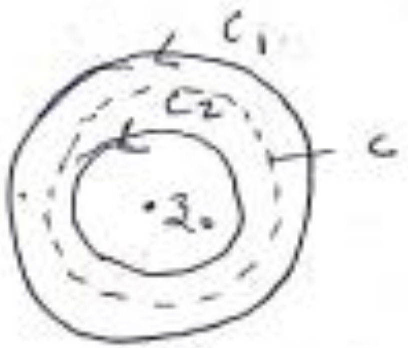
Let $f(z)$ be analytic in the annular domain bounded by two concentric circles C_1 & C_2 with centre z_0 and radii ρ_1 and ρ_2 $(\rho_1 > \rho_2)$ and let z be any point of D then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}.$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{(s-z_0)^{n+1}} ds$, $b_n = \frac{1}{2\pi i} \int_{C_2} (s-z_0)^{n-1} f(s) \, ds$

$n = 1, 2, 3, \ldots$

Proof of Laurent's theorem



Let z be any point of the domain then by Cauchy's Integral formula

$$f(z) = \frac{1}{2\pi i}\left[\int_{C_1} \frac{f(s)}{s-z}\,ds - \int_{C_2} \frac{f(s)}{s-z}\,ds\right] \qquad (1)$$

where $C_1 = \{z : |z-z_0| = \rho_1\}$ and $C_2 = \{z : |z-z_0| = \rho_2\}$

for the I^{st} integral in (1) we can proceed as the proof of Taylor's theorem and get

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s-z}\,ds = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where } a_n = \frac{f^{(n)}(z_0)}{n!} \qquad (2)$$

now consider the 2^{nd} Integral of (1)

for any point s on C_2, let

$$-\frac{1}{s-z} = \frac{1}{(z-z_0)-(s-z_0)} = \frac{1}{(z-z_0)\left[1-\frac{s-z_0}{z-z_0}\right]}$$

$$= \frac{1}{(z-z_0)}\left[1+\frac{s-z_0}{z-z_0}+\left(\frac{s-z_0}{z-z_0}\right)^2+\cdots + \left(\frac{s-z_0}{z-z_0}\right)^n \frac{1}{1-\frac{s-z_0}{z-z_0}}\right]$$

$$= \frac{1}{z-z_0}+\frac{s-z_0}{(z-z_0)^2}+\frac{(s-z_0)^2}{(z-z_0)^3}+\cdots\frac{(s-z_0)^{n-1}}{(z-z_0)^n}$$

$$+ \frac{(s-z_0)^n}{(z-z_0)^{n+1}\frac{(z-\cancel{z_0}-s+\cancel{z_0})}{(z-\cancel{z_0})}}$$

$$\therefore -\frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s-z}\,ds = \frac{1}{2\pi i\,(z-z_0)}\int_{C_2} f(s)\,ds + \frac{1}{2\pi i\,(z-z_0)^2}\int_{C_2}(s-z_0)^2 f(s)\,ds$$

$$+ \cdots \frac{1}{2\pi i (z-z_0)^n} \int_{C_2} (s-z_0)^{n-1} f(s)\, ds + S_n$$

where $S_n = \dfrac{1}{2\pi i (z-z_0)^n} \displaystyle\int_{C_2} \frac{(s-z_0)^n}{z-s} f(s)\, ds$

now let $b_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_2} (s-z_0)^{n-1} f(s)\, ds$ we have

$$-\frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s-z}\, ds = b_1 (z-z_0)^{-1} + b_2 (z-z_0)^{-2} + \cdots$$
$$+ b_n (z-z_0)^{-n} + S_n.$$

To show $S_n \to 0$ as $n \to \infty$,
we have $|z-z_0| = r$, $|s-z_0| = \rho_2$ and $\rho_2 < r$

$$|z-s| = |(z-z_0) - (s-z_0)| \geq |z-z_0| - |s-z_0|$$
$$= r - \rho_2$$

Hence

$$|S_n| \le \frac{1}{2\pi |(z-z_0)|^n} \int_{C_2} \frac{|s-z_0|^n}{\underbrace{|z-s|}_{\text{replaced by min value}}} |f(s)| |ds|$$

$$\le \frac{1}{2\pi r^n} \int_{C_2} \frac{\rho_2^{\,n}}{r-\rho_2} \cdot M_2 |ds| \quad \left\{ \begin{array}{l} \text{where} \\ M_2 = \max f(z) \\ \text{on } C_2 \end{array} \right\}$$

$$= \frac{1}{2\cancel{\pi} r^{n+1}} \cdot \frac{\rho_2^{\,n} M_2}{1-\frac{\rho_2}{r}} \cdot 2\cancel{\pi} \rho_2 = \frac{M_2}{1-\frac{\rho_2}{r}} \left(\frac{\rho_2}{r}\right)^{n+1}$$

$$\frac{\rho_2}{r} < 1 \quad \text{as} \quad r > \rho_2$$

$$\Rightarrow \frac{\rho_2}{r} \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow S_n \to 0 \quad \text{as } n \to \infty$$

Thus we get $-\frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s-z} ds = \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$ — (3)

using (2) & (3) in (1) we get

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

Remark

we can see that $b_n = a_{-n}$

hence the series expansion can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad \text{where } a_n = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z_0)^{n+1}}$$

and c is circle of radius ρ $\ni$

$$\rho_2 < \rho < \rho_1 .$$

Uniqueness Theorem

let there is another expression for $f(z)$ as

$$f(z) = \sum_{n=-\infty}^{\infty} p_n (z-z_0)^n \quad \rho_2 \le |z-z_0| < \rho_1$$

Then we have to show that it is identical with the Laurent series.

Proof. Let C be the circle $|z-z_0|=\rho$ where $\rho_2<\rho<\rho_1$. Then the coefficient a_n is the Laurent series expansion is given by

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z_0)^{n+1}}$$

$$= \frac{1}{2\pi i}\int_C \frac{1}{(s-z_0)^{n+1}} \sum_{m=-\infty}^{\infty} P_m (s-z_0)^m \, ds$$

$$= \frac{1}{2\pi i}\sum_{m=-\infty}^{\infty} P_m \int_C \frac{(s-z_0)^m}{(s-z_0)^{n+1}}\, ds$$

[term by term integratⁿ is possible as series is uniformly convergent on the given domain]

$$= \frac{1}{2\pi i}\sum_{m=-\infty}^{\infty} P_m \int_0^{2\pi} \frac{\rho^m e^{im\theta}}{\rho^{n+1} e^{i(n+1)\theta}}\, i\rho e^{i\theta}\, d\theta$$

$\{s-z_0=\rho e^{i\theta}\}$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} P_m \int_0^{2\pi} \rho^{m-n} e^{(m-n)i\theta} \, d\theta$$

when $m \neq n$, $\int_0^{2\pi} e^{(m-n)i\theta} d\theta = \left[\frac{e^{(m-n)i\theta}}{(m-n)i} \right]_0^{2\pi} = 0$ [show it]

when $m = n$, $\int_0^{2\pi} e^{(m-n)i\theta} d\theta = \int_0^{2\pi} d\theta = 2\pi$

Hence we get

$a_n = \frac{1}{2\pi} P_n \cdot 2\pi = P_n \Rightarrow$ Given series is identical with Laurent series.

Some Remarks on Laurent Series expansion

Remark:1 Every analytic function f in the annulus region $R_1 < |z| < R_2$ can be uniquely decomposed into a sum $f(z) = f_-(z) + f_+(z)$, where $f_+(z)$ is analytic for $|z| < R_2$, and $f_-(z)$ is analytic for $|z| > R_1$.

Remark-2 The expansion when $R_1 < 1 < R_2$.

Let f be analytic in some nbd, say $D = \{z : 1-\varepsilon < |z| < 1+\varepsilon\}$ $\varepsilon > 0$ of the unit circle $|z| = 1$. then for z in this nbd we get

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad a_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)\,dz}{z^{n+1}}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta \qquad \left\{ \begin{array}{l} z = e^{i\theta} \\ dz = i e^{i\theta} d\theta \end{array} \right.$$

In particular, let $f(e^{it}) = F(t)$ and $z = e^{it}$ we have

$$\boxed{F(t) = \sum_{n=-\infty}^{\infty} a_n e^{int} \quad \text{with} \quad a_n = \frac{1}{2\pi} \int_0^{2\pi} F(t) e^{-int}\, dt}$$

is the complex form.

Fourier series of F

Remark-3 If f is analytic at $z=0$ then the corresponding $f_-(z)=0$ {as z cannot be in denominator} and the Laurent series becomes the Taylor series about 0.

Ex-1. Show that when $0<|z|<4$

$$f(z)=\frac{1}{4z-z^2}=\sum_{n=0}^{\infty} z^{n-1}/4^{n+1}$$

Sol^n when $|z|<4$, we have $\frac{1}{4z-z^2}=\frac{1}{4z(1-z/4)}=\frac{1}{4z}\left(1-\frac{z}{4}\right)^{-1}$ $\left(\frac{z}{4}<1\right)$

$$\Rightarrow f(z)=\frac{1}{4z}\left[1+\frac{z}{4}+\left(\frac{z}{4}\right)^2+\cdots\right]$$

$$=\frac{1}{4z}+\frac{1}{4^2}+\frac{z}{4^3}+\frac{z^2}{4^4}+\cdots=\sum_{n=0}^{\infty}\frac{z^{n-1}}{4^{n+1}}$$

Ex. 2 Expand $\frac{1}{z(z^2-3z+2)}$ for the regions

(i) $0<|z|<1$, (ii) $1<|z|<2$ (iii) $|z|>2$

Sol^n Let $f(z)=\frac{1}{z(z^2-3z+2)}=\frac{1}{z(z-1)(z-2)}$

using partial fract^n, we get

$$f(z)=\frac{1}{2z}-\frac{1}{z-1}+\frac{1}{2(z-2)}$$

(i) $0<|z|<1$, $f(z)=\frac{1}{2z}+(1-\underbrace{z}_{<1})^{-1}-\frac{1}{4}(1-\underbrace{\frac{z}{2}}_{<1})^{-1}$

$$\Rightarrow f(z)=\frac{1}{2z}+\{1+z+z^2+z^3+\cdots\}-\frac{1}{4}\left\{1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right\}$$

$$=\frac{1}{2z}+\frac{3}{4}+\frac{7z}{8}+\frac{15}{16}z^2+\cdots$$

(ii) for $1<|z|<2$, $f(z)=\frac{1}{2z}-\frac{1}{z}\left(1-\underbrace{\frac{1}{z}}_{<1}\right)^{-1}-\frac{1}{4}\left(1-\underbrace{\frac{z}{2}}_{<1}\right)^{-1}$

$$\Rightarrow f(z)=\frac{1}{2z}-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)-\frac{1}{4}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

$$=\left(-\frac{1}{2z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots\right)-\frac{1}{4}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right]$$

(iii) $|z|>2$, $f(z)=\frac{1}{2z}-\frac{1}{z}\left(1-\underbrace{\frac{1}{z}}_{<1}\right)^{-1}+\frac{1}{2z}\left(1-\underbrace{\frac{2}{z}}_{<1}\right)^{-1}$

$$\Rightarrow f(z)=\frac{1}{2z}-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)+\frac{1}{2z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots\right)$$

$$=(2-1)\frac{1}{z^3}+(2^2-1)\frac{1}{z^4}+(2^3-1)\frac{1}{z^5}+\cdots$$

Thank You !