Edmond-Karp Algorithm (DAA, M.Tech + Ph.D.)

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Outline

- Edmond-Karp Algorithm
- Conclusion
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Edmonds-Karp Algorithm

- Edmonds-Karp algorithm is an implementation of the Ford-Fulkerson method for computing the maximum flow in a flow network in much mor optimized approach.

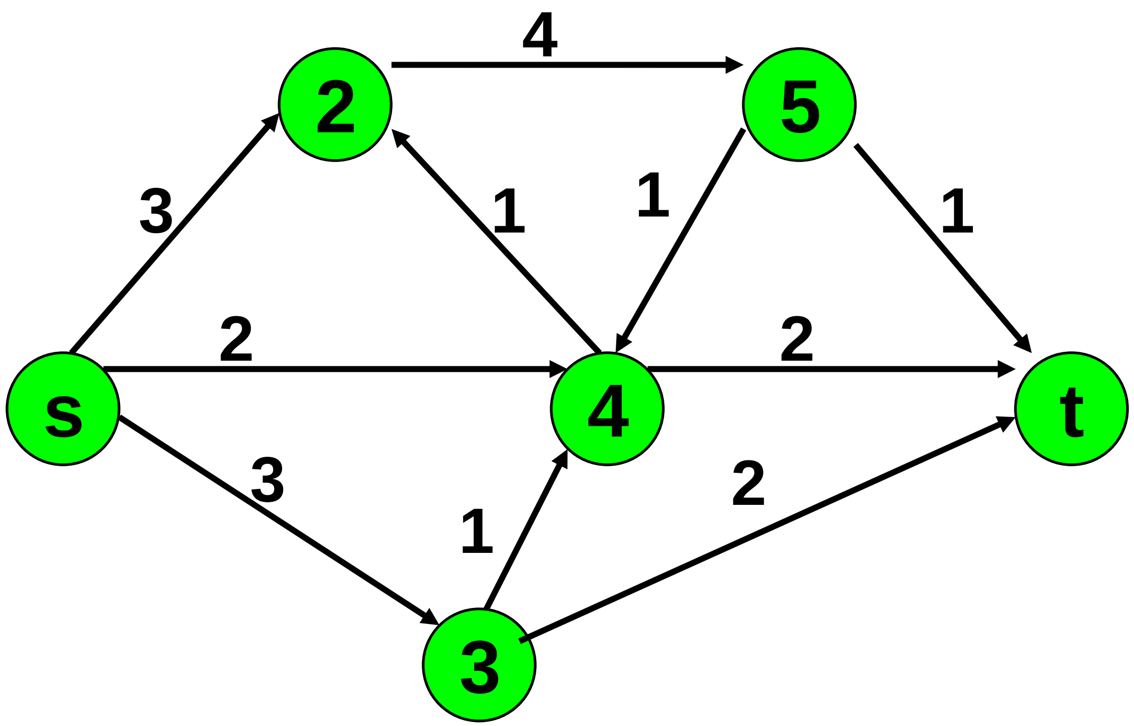
- Edmonds-Karp is identical to Ford-Fulkerson except for one very important trait. The search order of augmenting paths is well defined.

- The augmenting path is a shortest path from s to t in the residual graph (here, we count the number of edges for the shortest path).

Edmonds-Karp Algorithm

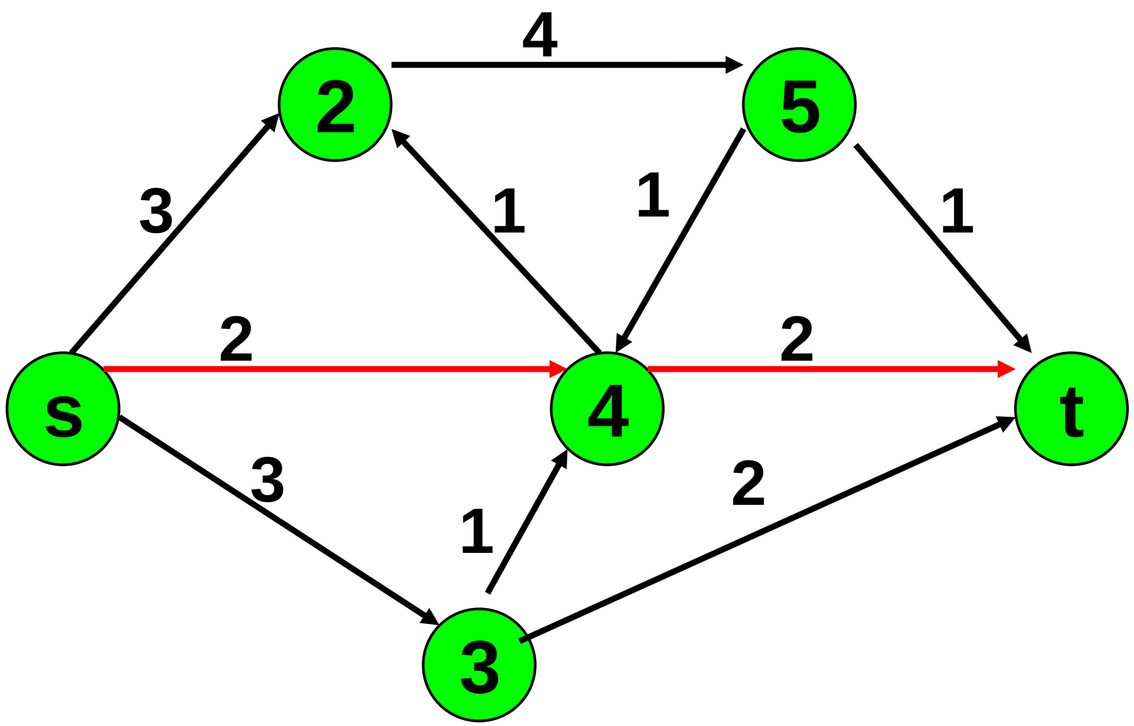
- It uses the breadth-first search approach
- This variant of Ford-Fulkerson algorithm runs in $O(nm^2)$

Ford-Fulkerson Max Flow



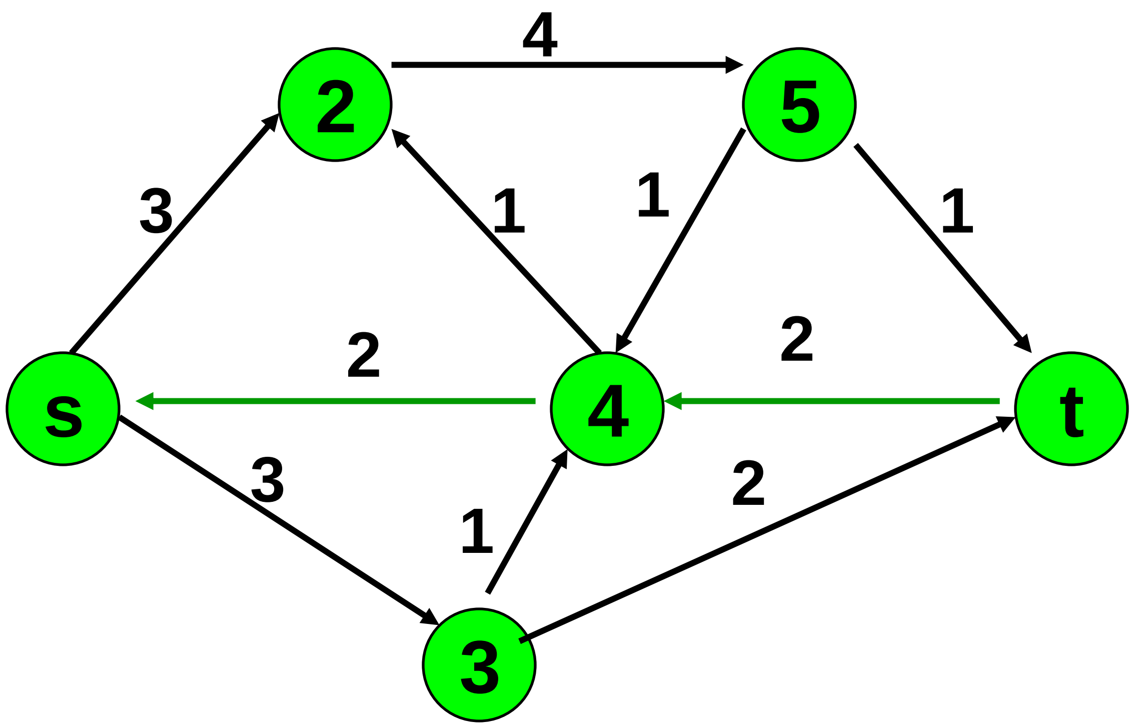
This is the original network.

Ford-Fulkerson Max Flow



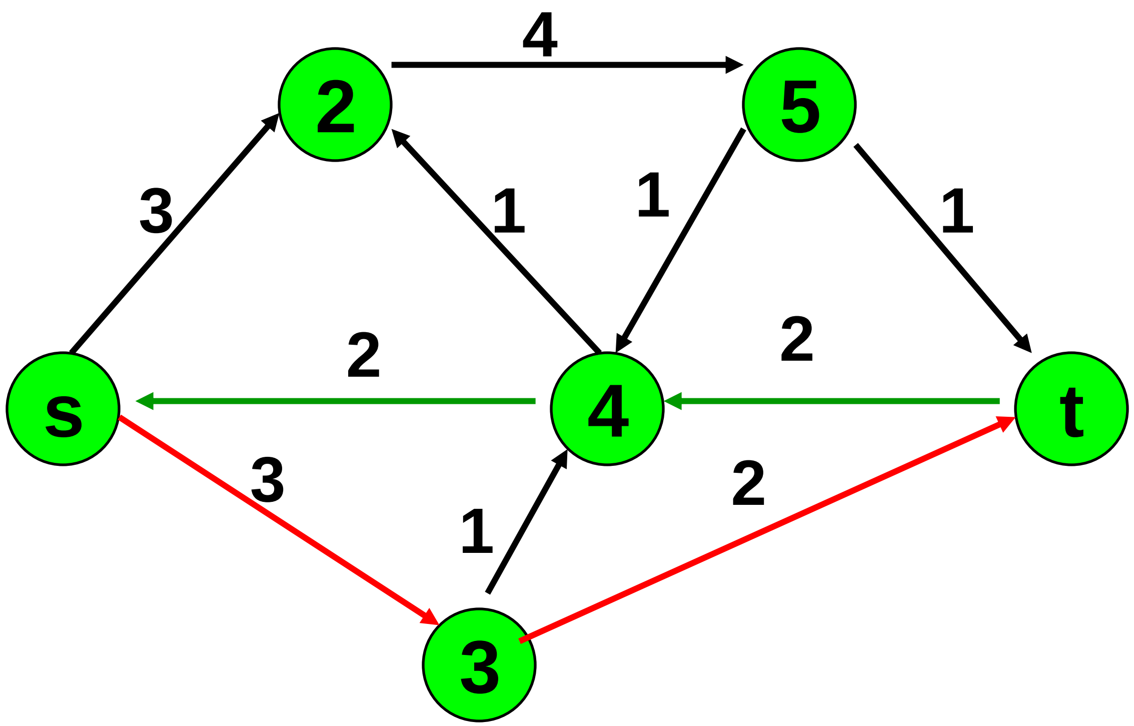
Choose a shortest path from *s* to *t*.

Ford-Fulkerson Max Flow



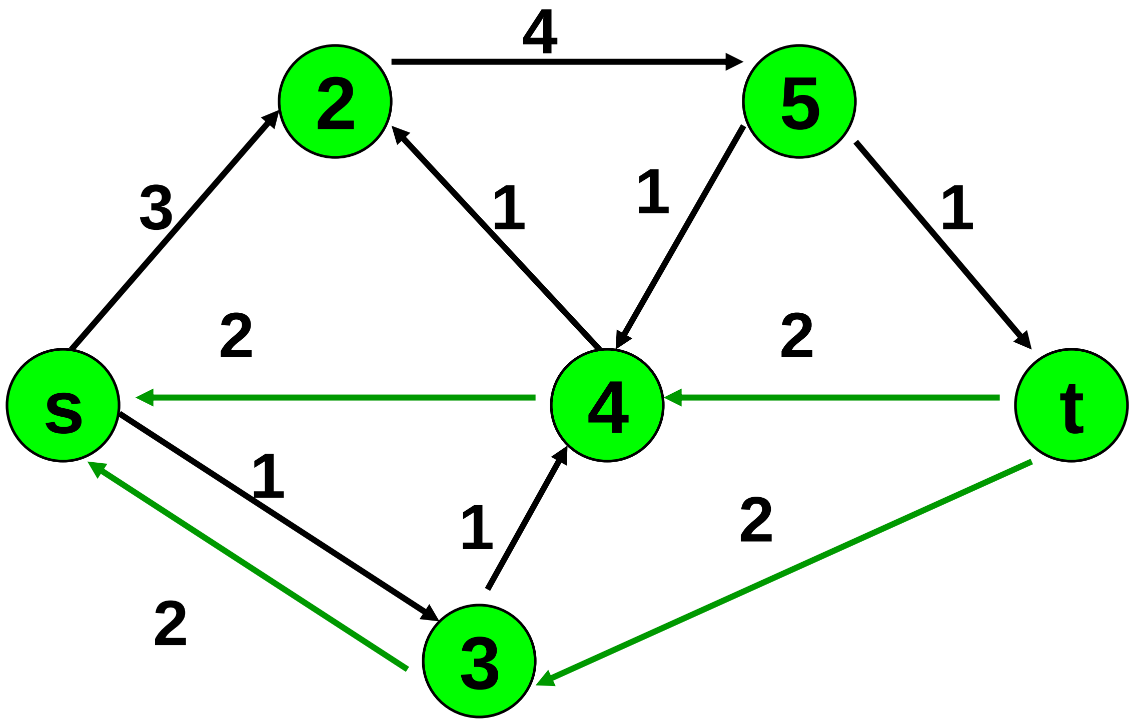
This is residual graph after the 1^{st} augmentation.

Ford-Fulkerson Max Flow



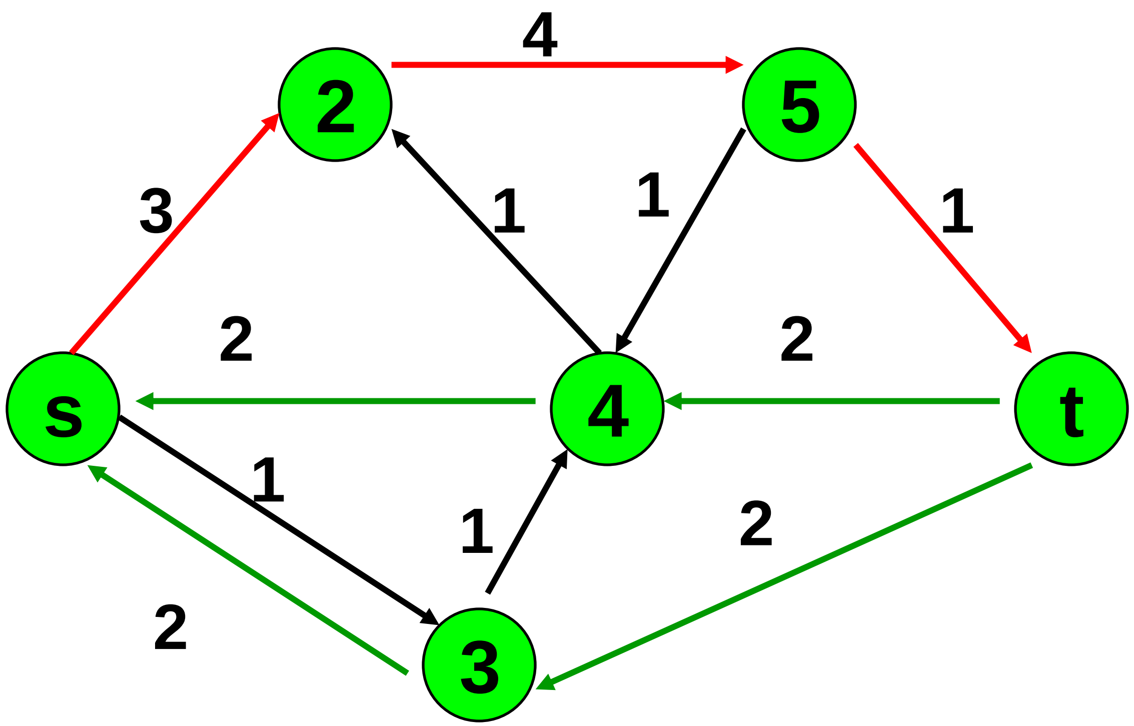
Choose a shortest path from s to t.

Ford-Fulkerson Max Flow



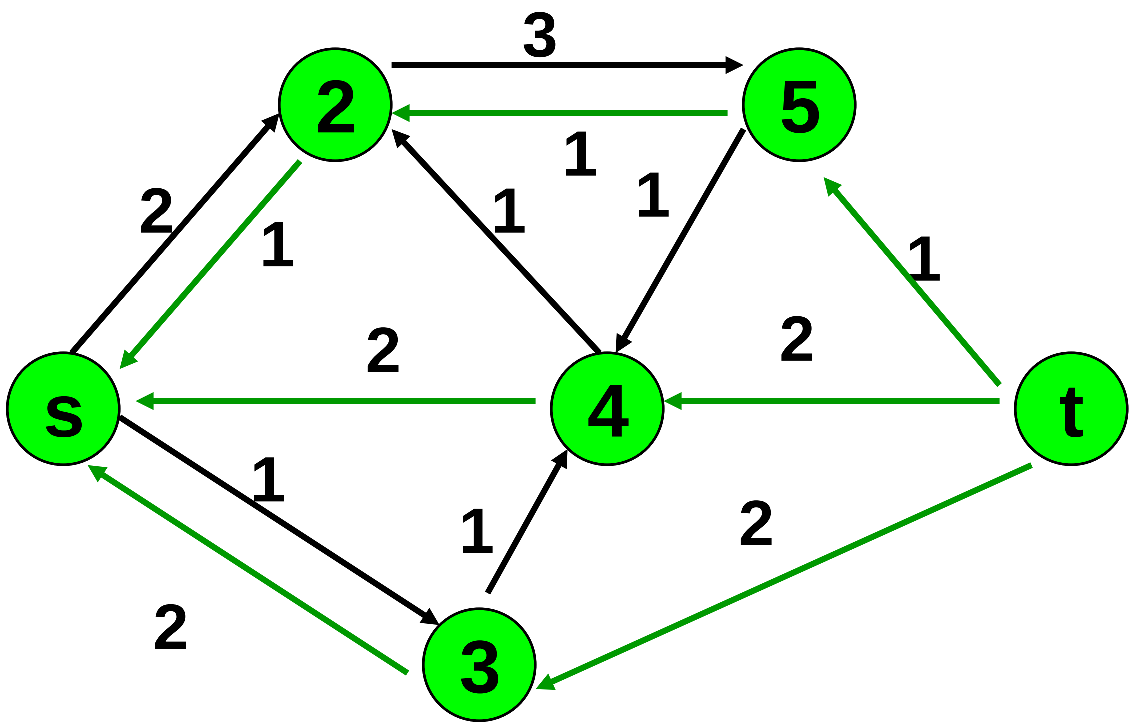
The residual graph after the 2^{nd} augmentation.

Ford-Fulkerson Max Flow



Choose a shortest path from s to t.

Ford-Fulkerson Max Flow



The residual graph after the 3rd augmentation.

Let $\delta_f(s, x)$ the shortest path distance from s to x in the residual network G_f, where each edge has unit distance.

Lemma

When Edmonds - Karp algorithm runs, $\delta_f(s, x)$ increases monotonically with each flow augmentation.

Proof

Denote $\delta_f(x) = \delta_f(s, x)$.

For contradiction, suppose flow f' is obtained from flow f through an augmentation and $\delta_{f'}(v) < \delta_f(v)$ for some node v.

W.l.g., assume $\delta_{f'}(v)$ is the smallest among such v, i.e.,

$$\delta_{f'}(u) < \delta_{f'}(v) \Rightarrow \delta_{f'}(u) \geq \delta_f(u).$$

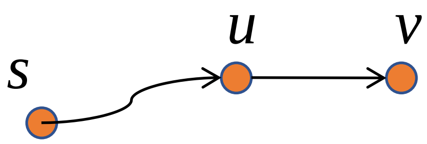
Suppose (u,v) is on the shortest path from s to v in $G_{f'}$.

Case 1. $(u,v) \in G_f$.

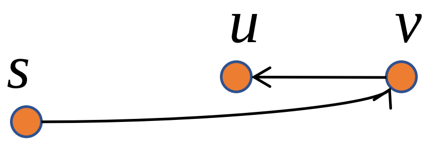
$\delta_f(v) \le \delta_f(u) + 1 \le \delta_{f'}(u) + 1 = \delta_{f'}(v), (\rightarrow\leftarrow)$.

Case 2. $(u,v) \notin G_f$. Then (v,u) must be on augmenting path in G_f.

$\delta_f(v) = \delta_f(u) - 1 \le \delta_{f'}(u) - 1 = \delta_{f'}(v) - 2 \le \delta_{f'}(v), (\rightarrow\leftarrow)$.



in $G_{f'}$



in G_f

(u,v) is critical in G_f if (u,v) has the smallest capacity in augmenting path p in G_f.

Lemma

Each (u,v) can be critical at most $(n+1)/2$ times.

Proof

Suppose (u,v) is critical in G_f. Then (u,v) will dispear in next residual graph. Before (u,v) appears again, (v,u) has to appear in augmenting path of a residual graph $G_{f'}$. Thus, we have

$\delta_{f'}(u) = \delta_{f'}(v) + 1.$

Since $\delta_f(v) \le \delta_{f'}(v)$, we have

$\delta_{f'}(u) = \delta_{f'}(v) + 1 \ge \delta_f(v) + 1 = \delta_f(u) + 2.$

Theorem

Edmonds-Karp algorithm runs in time $O(|V|\cdot|E|^2)$.

Proof

In every augmentation, there exists an arc critical.

Since each arc can be critical at most $(n+1)/2$ times,

there are at most $O(|V|\cdot|E|)$ augmentations.

In each augmentation, finding the shortest path

takes $O(|E|)$ time.

References

1. Cormen, Thomas H., Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to algorithms*. MIT press, 2009.
2. Cormen, Thomas H., Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. "Introduction to algorithms second edition." *The Knuth-Morris-Pratt Algorithm, year* (2001).
3. Seaver, Nick. "Knowing algorithms." (2014): 1441587647177.

Thank You