Basic Probability Theory-II

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Axiomatic approach to probability:

Probability function: $P(A)$ is the probability function defined on a σ-field $\mathcal{B}$ of events if the following properties or axioms hold

1. For each $A \in \mathcal{B}$, $P(A)$ is defined, is real and $1 \geq P(A) \geq 0$
2. $P(S) = 1$
3. If $\{A_n\}$ is any finite or infinite sequence of disjoint events in $\mathcal{B}$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$

The triplet $(S, \mathcal{B}, P)$ is often called the probability space.

Note: σ-field $\mathcal{B}$; $S = \{a, b, c, d\}$.

Let $A = \{a, b\}$, $A^c = \{c, d\}$

$\mathcal{B} = \{\phi, S, A, A^c\}$ is σ-field $\mathcal{B}$.

Algebra of events: For events A, B, C

(i) $A \cup B = \{\omega \in S : \omega \in A \text{ or } \omega \in B\}$

(ii) $A \cap B = \{\omega \in S : \omega \in A \text{ and } \omega \in B\}$

(iii) $\bar{A} = \{\omega \in S : \omega \notin A\}$

(iv) $A - B = \{\omega \in S : \omega \in A \text{ but } \omega \notin B\}$

(v) $A \subset B \Rightarrow$ for every $\omega \in A, \omega \in B$

(vi) A and B disjoint (mutually exclusive) $\Rightarrow A \cap B = \phi$

(vii) $A \cup B$ can be denoted by $A + B$ if A and B are disjoint.

(viii) $A \Delta B$ denotes those ω belonging to exactly one of A and B i.e.

$$A \Delta B = \bar{A}B \cup A\bar{B} = \bar{A}B + A\bar{B} \text{ (disjoint events)}$$

(ix) Only A occurs $\Leftrightarrow A \cap \bar{B} \cap \bar{C}$

(x) Both A and B, but not C occur $\Leftrightarrow A \cap B \cap \bar{C}$

(xi) All three events occur $\Leftrightarrow A \cap B \cap C$

(xii) At least one occurs $\Leftrightarrow A \cup B \cup C$

(xiii) At least two occurs $\Leftrightarrow (A\cap B\cap \bar{C}) \cup (A\cap \bar{B} \cap C) \cup (\bar{A}\cap B\cap C) \cup (A\cap B\cap C)$

(xiv) One and ~~more~~ no more occurs $\Leftrightarrow (A\cap \bar{B}\cap \bar{C}) \cup (\bar{A}\cap B\cap \bar{C}) \cup (\bar{A}\cap \bar{B}\cap C)$.

(xv) Two and no more occur $\Leftrightarrow (A\cap B\cap \bar{C}) \cup (\bar{A}\cap B\cap C) \cup (A\cap \bar{B}\cap C)$

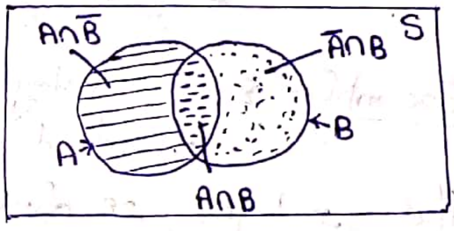
(xvi) None occurs $\Leftrightarrow \overline{(A\cup B\cup C)}$ or $(\bar{A}\cap \bar{B}\cap \bar{C})$.

<u>Addition theorem of probability</u>: If A and B are any two events (Subset of sample spaces) and are not disjoint, then

$$P(A\cup B) = P(A) + P(B) - P(A\cap B).$$

<u>Proof</u>: We have

$$A\cup B = A \cup (\bar{A}\cap B)$$



where A and $\bar{A}\cap B$ are mutually disjoint (from the Venn diagram)

$$\therefore P(A\cup B) = P[A\cup(\bar{A}\cap B)]$$

$$= P(A) + P(\bar{A}\cap B) \quad \text{(By axiom (3))}$$

$$\cancel{P(A) + P(B) - P(A\cap B)}$$

$$= P(A) + [P(\bar{A}\cap B) + P(A\cap B)] - P(A\cap B)$$

$$= P(A) + [P\{(\bar{A}\cap B)\cup(A\cap B)\}] - P(A\cap B)$$

$$= P(A) + P(B) - P(A\cap B)$$

Proved

<u>Example-1</u>- Two dice are tossed. Find the probability of getting 'An even number on the first die or a total of 8'.

<u>Solution</u>- Sample space S ~~is~~ given by a random toss of two dice is

$$S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}. \quad n(S) = 6\times 6 = 36$$

Let us define the events

A: Getting an even number on the first dice.

B: the sum of the points obtained on the two dice 8.

Therefore, $A = \{2,4,6\} \times \{1,2,3,4,5,6\} \Rightarrow n(A) = 3 \times 6 = 18$

$B = \{(2,6),(3,5),(4,4),(5,3),(6,2)\} \Rightarrow n(B) = 5.$

and $A \cap B = \{(2,6),(4,4),(6,2)\} \Rightarrow n(A \cap B) = 3.$

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}, \quad P(B) = \frac{5}{36}, \quad P(A \cap B) = \frac{3}{36}$

Hence required probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2} + \frac{5}{36} - \frac{3}{36} = \frac{20}{36} = \frac{5}{9}$

Example-2- An integer is chosen at random from two hundred digits. What is the probability that the integer is divisible by 6 or 8?

Solution- The sample space of the random experiment

$S = \{1,2,3,\ldots,200\} \Rightarrow n(S) = 200$

Let us define the events

A: integer chosen is divisible by 6

$A = \{6,12,18,\ldots 198\} \quad n(A) = \frac{198}{6} = 33, \quad P(A) = \frac{33}{200}$

B: Integer chosen is divisible by 8

$B = \{8,16,24,\ldots,200\} = n(A) = \frac{200}{8} = 25, \; P(B) = \frac{25}{200}$

$A \cap B$: A number divisible by both 6 and 8 is divisible by 24

$A \cap B = \{24,48,72,\ldots 192\}, \; n(A \cap B) = \frac{192}{24} = 8, \; P(A \cap B) = \frac{8}{200}$

Hence required probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{33}{200} + \frac{25}{200} - \frac{8}{200} = \frac{50}{200} = \frac{1}{4}$

Example - 3 - The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then find $P(\bar{A}) + P(\bar{B})$.

Solution - we have

P(at least one of the events A and B occurs) = 0.6

i.e. $P(A \cup B) = 0.6$

P(A and B occur simultaneously) = 0.2

i.e. $P(A \cap B) = 0.2$.

We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$0.6 = 1 - P(\bar{A}) + 1 - P(\bar{B}) - 0.2$$

$$P(\bar{A}) + P(\bar{B}) = 1.2$$

Conditional Probability: Let A and B be two events associated with a random experiment. Then the probability of occurrence of event A under the condition that B has already occurred and $P(B) \neq 0$, is the conditional probability and it is denoted by $P(A|B)$.

Thus we have

$P(A/B)$ = Conditional probability of occurrence of event A under the condition that event B has occurred already $(P(B) \neq 0)$

Similarly

$P(B/A)$ = Conditional probability of occurrence of event B under the condition that event A has already occurred $(P(A) \neq 0)$.

$$P(A|B) = \frac{n(A \cap B)}{n(B)}, \quad P(B/A) = \frac{n(A \cap B)}{n(A)}.$$

Example - A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the probability that the number 4 has appeared at least once?

Solution - Let A = Event of appearing those numbers whose sum is 6.

B = Event that number 4 has appeared at least once.

i.e. $A = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$B = \{(2,4), (4,2), (1,4), (4,1), (3,4), (4,3), (4,4), (4,5), (5,4), (4,6), (6,4)\}$

$A \cap B = \{(2,4), (4,2)\}$

Required probability $P(B/A) = \dfrac{n(A \cap B)}{n(A)} = \dfrac{2}{5}$.

Multiplication theorem of probability: If A and B are two events associated with a random experiment, then

$$P(A \cap B) = P(A).\ P(B/A) \quad \text{if } P(A) \neq 0$$
$$= P(B).\ P(A/B) \quad \text{if } P(B) \neq 0.$$

Proof - Let S be the sample space

$$P(A) = \frac{n(A)}{n(S)},\quad P(B) = \frac{n(B)}{n(S)},\quad P(A \cap B) = \frac{n(A \cap B)}{n(S)}.$$

For the conditional event A/B, the favourable elementary events must be one of the sample points of B i.e. for the event A/B, the sample space is B and out of the $n(B)$ sample points $n(A \cap B)$ are favourable to event A.

Hence $P(A|B) = \dfrac{n(A\cap B)}{n(B)}$

$$P(A\cap B) = \frac{n(A\cap B)}{n(S)}$$

$$= \frac{n(A\cap B)}{n(B)} \cdot \frac{n(B)}{n(S)}$$

$$= P(A|B) \cdot P(B)$$

$\therefore$ $P(A\cap B) = P(A|B) \cdot P(B)$.

Similarly $P(A\cap B) = P(A) \cdot P(B/A)$.

Proved.

Example - A bag contain 10 white and 15 black balls. Two balls are drawn in succession without replacement. What is the probability that first is white and 2^{nd} is black?

Solution - Consider the following events:

10W
15B

A: getting a white ball in 1^{st} drawn

B: getting a black ball in 2^{nd} drawn

Required probability $P(A\cap B) = P(A) \cdot P(B/A)$ --- (1)

$$P(A) = \frac{10}{25} = \frac{2}{5}, \quad P(B/A) = \frac{15}{24} = \frac{5}{8}.$$

$$\therefore \; P(A\cap B) = \frac{2}{5} \times \frac{5}{8} = \frac{1}{4} \quad (\text{By } (1))$$

Independent Events: Events are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence or non occurrence of the other.

Note - If A and B are independents events, then

$P(B/A) = P(B)$.

By multiplication theorem of probability, we have

$P(A \cap B) = P(A) \cdot P(B/A)$

$P(A \cap B) = P(A) \cdot P(B)$

* If $P(A \cap B) \neq P(A) \cdot P(B) \Rightarrow A$ & B are dependent events.
* If A, B & C are independent events, then

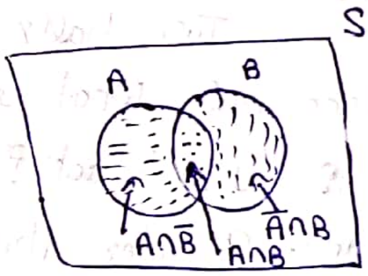
$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C).$$

<u>Theorem</u> - If A and B are independent events, then prove that

(1) $\bar{A}$ and B are independent events.

(2) A and $\bar{B}$ are independent events.

(3) $\bar{A}$ and $\bar{B}$ are independent events



<u>Proof</u> (1)

By Venn-diagram

$(A \cap B) \cup (\bar{A} \cap B) = B$

$P((A \cap B) \cup (\bar{A} \cap B)) = P(B)$

$P(A \cap B) + P(\bar{A} \cap B) = P(B)$ ($\because A \cap B$ & $\bar{A} \cap B$ are mutually exclusive events)

$P(A) \cdot P(B) + P(\bar{A} \cap B) = P(B)$ ($\because A$ & B are independent events)

$P(\bar{A} \cap B) = P(B)\{1 - P(A)\}$

$P(\bar{A} \cap B) = P(B)\, P(\bar{A})$

$\therefore$ $\bar{A}$ and B are independent events

Proved

Similarly prove (2) & (3).

<u>Example</u> - Two dice are thrown. Find the probability of getting and odd number on the first die and a multiple of 3 on the other.

Solution - Consider the following events

A: Getting an odd number on the 1^{st} die

B: getting a multiple of 3 on the 2^{nd} die.

i.e. $A = \{1, 3, 5\}$ and $B = \{3, 6\}$

$\therefore P(A) = \frac{3}{6} = \frac{1}{2}$; $P(B) = \frac{2}{6} = \frac{1}{3}$.

Req. probability $P(A \cap B) = P(A) \cdot P(B)$ ($\because A$ & B are independent events)

$= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

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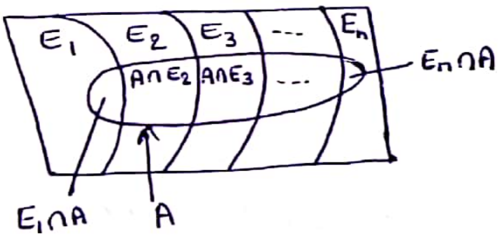
Bayes' theorem: Let S be the sample space and let $E_1, E_2, \dots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any events which occurs with E_1 or E_2 or .. or E_n, then

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)}, \quad i = 1, 2, \dots n.$$

Proof: Since $E_1, E_2, \dots E_n$ are n mutually exclusive and exhaustive events we have



$$S = E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n$$

& $E_i \cap E_j = \phi$ where $i \neq j$.

$$A = A \cap S$$
$$= A \cap (E_1 \cup E_2 \cup \dots \cup E_n)$$
$$= (A \cap E_1) \cup (A \cap E_2) \cup \dots \cup (A \cap E_n)$$

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \dots + P(A \cap E_n)$$

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i) \quad \text{---- (1)}$$

$\because$ $P(A \cap E_i) = P(E_i) \cdot P(A/E_i)$

from (1)

$$P(A) = \sum P(E_i)\,P(A/E_i)$$

By multiplication theorem, we have

$$P(A \cap E_i) = P(A) \cdot P(E_i/A)$$

i.e. $\cancel{P(E_i/A) = \frac{P(A \cap E_i)}{P(A)}}$ $\quad P(E_i/A) = \frac{P(A \cap E_i)}{P(A)}$

$\cancel{P(E_i/A}$ $\quad P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)}, \quad i = 1, 2, \dots n$

Proved.

Example - There are two bags A and B. A contains n white and 2 black balls and B contains 2 white and n black balls. One of the two bags is selected at random and two balls are drawn from it without replacement. If both the balls drawn are white and the probability that the bag A was used to draw the balls is 6/7, find the value of n.

Solution - Let

E_1: the event that bag A is selected

E_2: the event that bag B is selected

E: the event that two balls drawn are white.

$\therefore\ P(E_1) = \frac{1}{2},\ P(E_2) = \frac{1}{2}$

$$P(E/E_1) = \frac{{}^{n}C_2}{{}^{n+2}C_2} = \frac{n(n-1)}{(n+2)(n+1)}$$

and $P(E/E_2) = \frac{{}^{n}C_2}{{}^{n+2}C_2} = \frac{2}{(n+2)(n+1)}$

Using Bayes theorem the probability that the two white balls drawn are from the bag A is given

$$P(E_1|E) = \frac{P(E_1).P(E/E_1)}{P(E_1)P(E/E_1) + P(E_2)P(E/E_2)} = \frac{6}{7} \quad \text{(given)}$$

$$\Rightarrow \frac{\frac{1}{2}\cdot\frac{n(n-1)}{(n+2)(n+1)}}{\frac{1}{2}\frac{n.(n-1)}{(n+2)(n+1)} + \frac{1}{2}\cdot\frac{2}{(n+2)(n+1)}} = \frac{6}{7}$$

$$\Rightarrow \frac{n(n-1)}{n(n-1)+2} = \frac{6}{7} \quad \Rightarrow 7n(n-1) = 6n(n-1) + 12$$

$$n^2 - n - 12 = 0 \Rightarrow n = 4, -3$$

Since n can not be negative we have $n = 4$.

Question - 1 - A card from a pack of 52 card is lost. From the remaining cards of the pack, two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart.

X Question - 2 - The probability of X, Y and Z becoming managers are $\frac{4}{9}$, $\frac{2}{9}$ and $\frac{1}{3}$ respectively. The probability that the Bonus scheme will be introduced if X, Y and Z becomes managers are $\frac{3}{10}$, $\frac{1}{2}$ and $\frac{4}{5}$ respectively. (i) What is the probability that Bonus scheme will be introduced, and (ii) if the Bonus scheme has been introduced, what is the probability that the manager appointed was X?

Question - 3 - A and B are two weak students of statistics and their chances of solving a problem in statistics correctly are $\frac{1}{6}$ and $\frac{1}{8}$ respectively. If the probability of their making a common error is $\frac{1}{525}$ and they obtain the same answer, find the probability that their answer is correct.

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THANK YOU