Operations for fuzzy sets: union, intersection, complement

- Given two fuzzy sets $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$ and $\tilde{B} = \{(x, \mu_{\tilde{B}}(x)) | x \in X\}$ over the same universe of discourse X, we can define operations of union, intersection and complement. We define:
- the *union* of the fuzzy sets $\tilde{A}$ si $\tilde{B}$ as the fuzzy set $\tilde{C} = \tilde{A} \cup \tilde{B}$, given by $\tilde{C} = \{(x, \mu_{\tilde{C}}(x)) | x \in X\}$, where

$$\mu_{\tilde{C}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

- the *intersection* of the fuzzy sets $\tilde{A}$ and $\tilde{B}$ as the fuzzy set $\tilde{D} = \tilde{A} \cap \tilde{B}$, given by $\tilde{D} = \{(x, \mu_{\tilde{D}}(x)) | x \in X\}$, where

$$\mu_{\tilde{D}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

- the *complement* of $\tilde{A}$ in X as the fuzzy set $\tilde{E} = \mathbb{C}_{\tilde{A}} X$ given by $\tilde{E} = \{(x, \mu_{\tilde{E}}(x)) | x \in X\}$, where

$$\mu_{\tilde{E}}(x) = 1 - \mu_{\tilde{A}}(x)$$

Operations with fuzzy sets: inclusion, equality

- *inclusion* of fuzzy sets: given two fuzzy sets $\tilde{A}$ and $\tilde{B}$ included in X, the inclusion $\tilde{A} \subseteq \tilde{B}$ takes place iff $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$, $(\forall) x \in X$
- *equality* of two fuzzy sets: two fuzzy sets $\tilde{A}$ and $\tilde{B}$ included in X are equals iff $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$, $(\forall) x \in X$
- Equivalently, two fuzzy sets $\tilde{A}$ and $\tilde{B}$ included in X are equals iff $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$

Operations with fuzzy sets: examples

1. Determine the union and intersection of the fuzzy sets $\tilde{A} =$ "comfortable house for a 4 persons - family" and $\tilde{B} =$ "small house", where
   $\tilde{A} = \{(1, 0.1), (2, 0.5), (3, 0.8), (4, 1.0), (5, 0.7), (6, 0.2)\}$ and
   $\tilde{B} = \{(1, 1), (2, 0.8), (3, 0.4), (4, 0.1)\}$:
   $\tilde{A} \cup \tilde{B} = \{(1, \max(0.1, 1)), (2, \max(0.5, 0.8)), (3, \max(0.8, 0.4)),$
   $(4, \max(1, 0.1)), (5, \max(0.7, 0)), (6, \max(0.2, 0)\} =$
   $\{(1, 1), (2, 0.8), (3, 0.8), (4, 1), (5, 0.7), (6, 0.2)\}$
   $\tilde{A} \cap \tilde{B} = \{(1, \min(0.1, 1)), (2, \min(0.5, 0.8)), (3, \min(0.8, 0.4)),$
   $(4, \min(1, 0.1)), (5, \min(0.7, 0)), (6, \min(0.2, 0)\} =$
   $\{(1, 0.1), (2, 0.5), (3, 0.4), (4, 0.1), (5, 0), (6, 0)\}$
   $\tilde{A} \cup \tilde{B}$ can be read as "comfortable house for a 4 persons - family or small", and $\tilde{A} \cap \tilde{B}$ as "comfortable house for a 4 persons - family and small"

Operations with fuzzy sets: examples

2. Determine $\mathbb{C}_{\tilde{A}}X$, where $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$:
   ("non-comfortable house for a 4 persons - family")
   $\mathbb{C}_{\tilde{A}}X = \{(1, 1-0.1), (2, 1-0.5), (3, 1-0.8), (4, 1-1), (5, 1-0.7), (6, 1-0.2), (7, 1-0), (8, 1-0), (9, 1-0), (10, 1-0)\} = \{(1, 0.9), (2, 0.5), (3, 0.2), (4, 0), (5, 0.3), (6, 0.8), (7, 1), (8, 1), (9, 1), (10, 1)\}$
3. Determine the union and intersection of the fuzzy sets $\tilde{A} =$ "real numbers close to 10" and $\tilde{B} =$ "real number considerably larger than 11".
   - Analytically: $\tilde{C} = \tilde{A} \cup \tilde{B}$ si $\tilde{D} = \tilde{A} \cap \tilde{B}$, where
     $\mu_{\tilde{C}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$,
     $\mu_{\tilde{D}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$
     ...
   - Graphically: (more suited in this case), in the next slides:

Example of operations with fuzzy sets: union

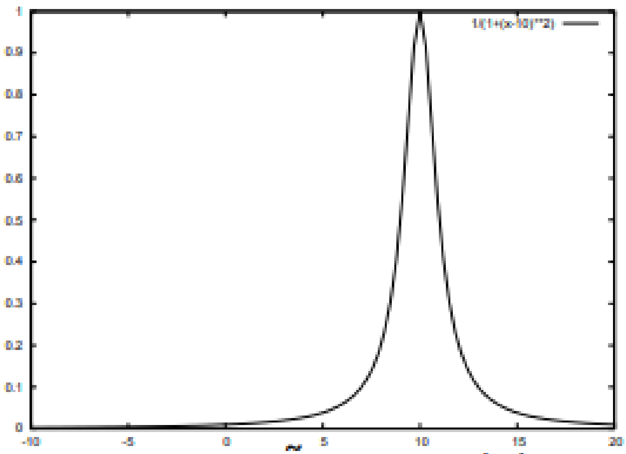

Figure 3: $\tilde{A}$ with $\mu_{\tilde{A}}(x)$

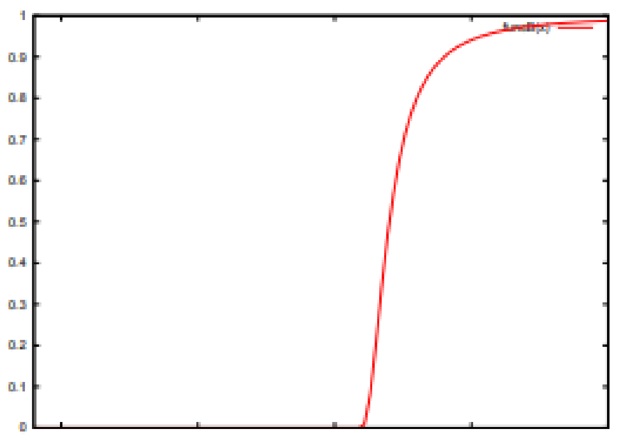

Figure 4: $\tilde{B}$ with $\mu_{\tilde{B}}(x)$

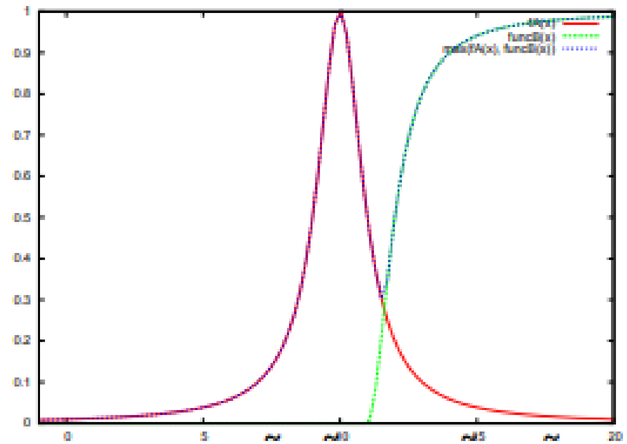

Figure 5: $\tilde{A}$, $\tilde{B}$ and $\tilde{A} \cup \tilde{B}$

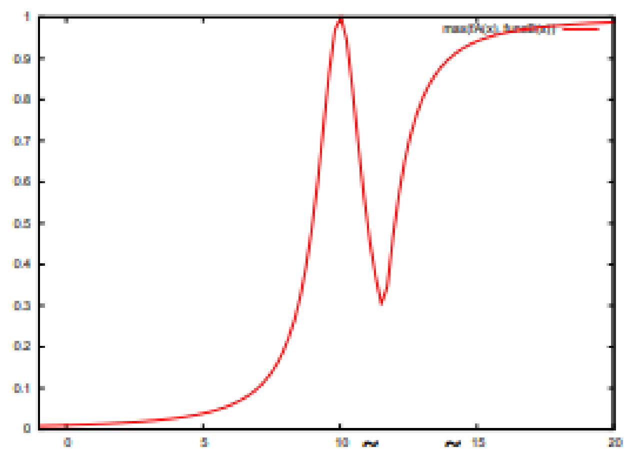

Figure 6: $\tilde{A} \cup \tilde{B}$

Example of operations with fuzzy sets: intersection

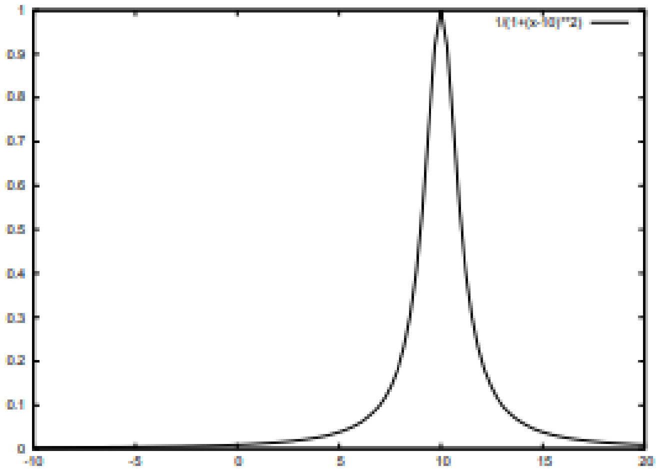

Figure 7: $\mu_{\tilde{A}}(x)$

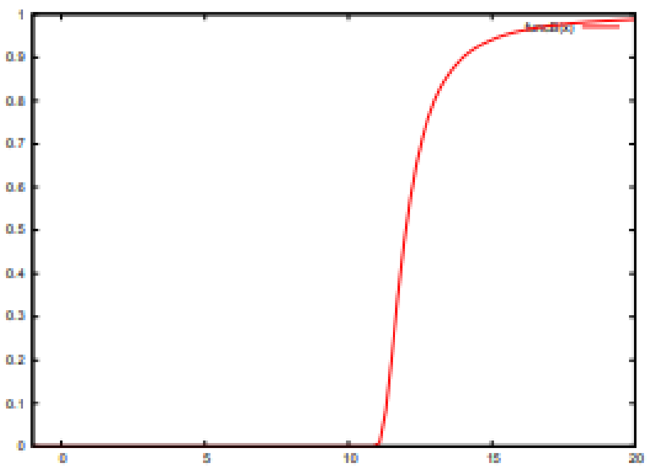

Figure 8: $\mu_{\tilde{B}}(x)$

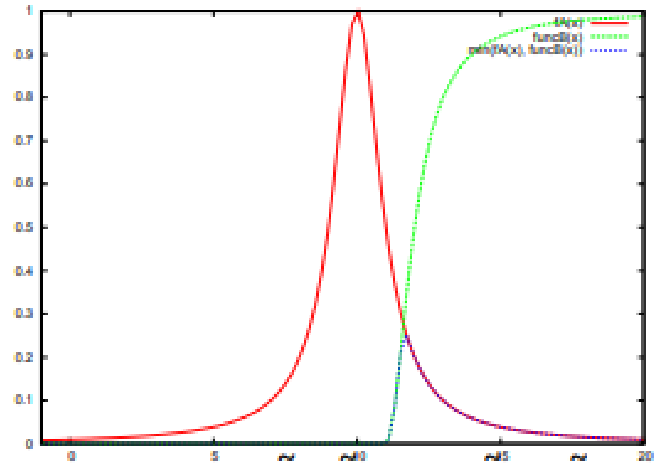

Figure 9: $\tilde{A}$, $\tilde{B}$ and $\tilde{A} \cap \tilde{B}$

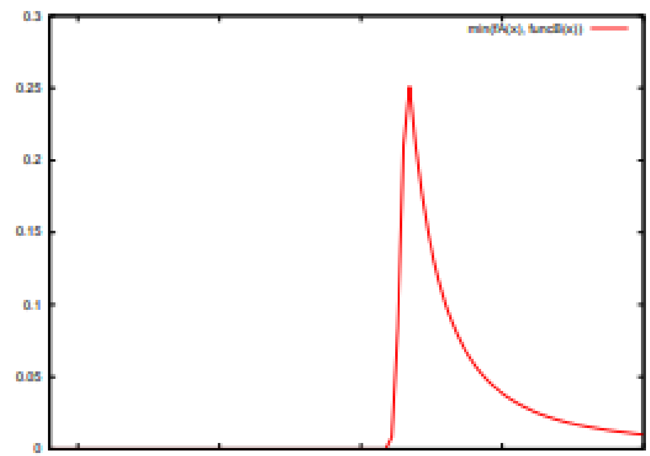

Figure 10: $\tilde{A} \cap \tilde{B}$ (detail)

Properties of the operations with crisp sets and fuzzy sets

For crisp sets in the universe of discourse X the following properties are true (after [NR74]):

1. Commutativity:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

2. Associativity:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

3. Distributivity:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4. Idempotency:

$$A \cup A = A$$

$$A \cap A = A$$

Properties of the operations with crisp sets and fuzzy sets

5. Identity:

$$A \cup \emptyset = \emptyset \cup A = A$$

$$A \cup X = X \cup A = X$$

$$A \cap \emptyset = \emptyset \cap A = \emptyset$$

$$A \cap X = X \cap A = A$$

6. Transitivity: if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
7. Involution: $\overline{\overline{A}} = A$, where $\overline{A} = \mathbb{C}_A X$
8. De Morgan:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Properties of the operations with crisp sets and fuzzy sets

9. Absorption:

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

10. Excluded middle laws *(excluded middle laws)*:

$$A \cup \overline{A} = X$$

$$A \cap \overline{A} = \emptyset$$

- Proprieties 1–9 hold for fuzzy sets, too, but NOT the property 10.
- Some researchers consider this fact (non-fulfillment of the excluded middle laws) as being the main characteristic of fuzzy sets.

Fuzzy Relations

Definition

Given the universes of discourse X and Y, a fuzzy relation $\tilde{R}$ in $X \times Y$ is defined as the set
$\tilde{R} = \{((x, y), \mu_{\tilde{R}}(x, y)) \mid (x, y) \in X \times Y\}$, where
$\mu_{\tilde{R}}(x, y) : X \times Y \to [0, 1]$

Examples of fuzzy relations

1. For $X = Y = \mathbb{R}$, we define the continuous fuzzy relation "x considerably larger than y" :
$$\mu_{\tilde{R}}(x, y) = \begin{cases} 0, \text{ if } x \leq y \\ \frac{|x-y|}{10 \cdot |y|}, \text{ if } y < x \leq 11 \cdot y \\ 1, \text{ if } x > 11 \cdot y \end{cases}$$
2. The fuzzy relation "$x \gg y$" could be defined also as:
$$\mu_{\tilde{R}}(x, y) = \begin{cases} 0, \text{ if } x \leq y \\ \frac{(x-y)^2}{1+(x-y)^2}, \text{ if } x > y \end{cases}$$

Examples of fuzzy relations

3. For the discrete fuzzy sets $X = \{x_1, x_2\}, Y = \{y_1, y_2, y_3\}$, the fuzzy relation $\tilde{R}$ "$x \gg y$" can be expressed by the matrix:

	y_1	y_2	y_3
x_1	0.5	1	0
x_2	0.7	0.2	0.1

Table 1: Fuzzy relation $\tilde{R}$