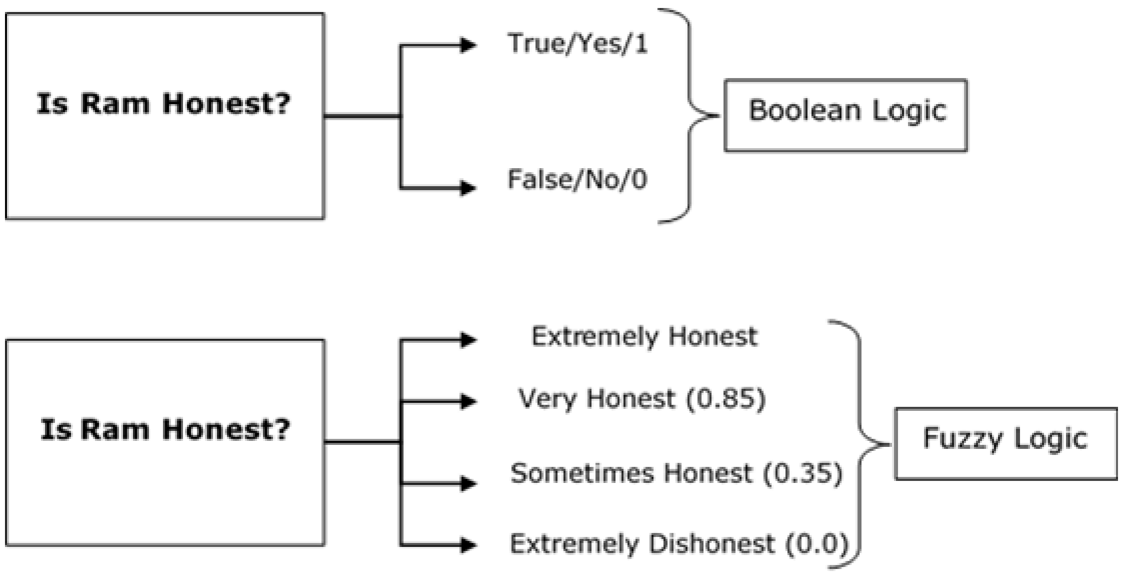
What is Fuzzy Logic?

Fuzzy Logic resembles the human decision-making methodology.
It deals with vague and imprecise information. T
his is gross oversimplification of the real-world problems and based on degrees of truth rather than usual true/false or 1/0 like Boolean logic.



Fuzzy Logic - Classical Set Theory

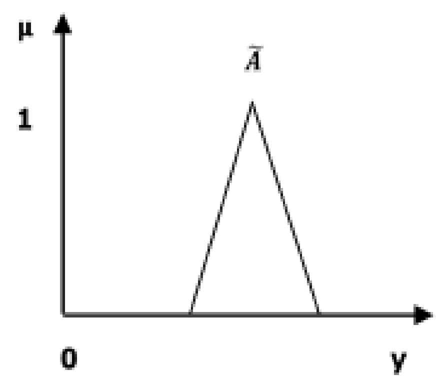
- A set is an unordered collection of different elements.
- It can be written explicitly by listing its elements using the set bracket.
- If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

Example

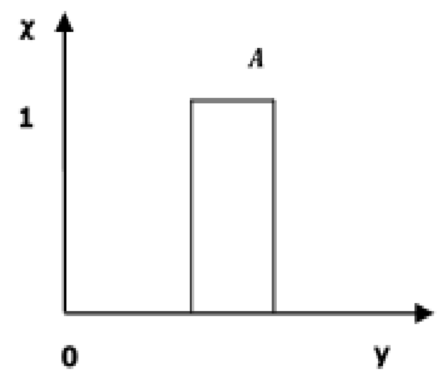
- A set of all positive integers.
- A set of all the planets in the solar system.
- A set of all the states in India.
- A set of all the lowercase letters of the alphabet.

Fuzzy Logic - Set Theory

- Fuzzy sets can be considered as an extension and gross oversimplification of classical sets.
- Basically it allows partial membership which means that it contain elements that have varying degrees of membership in the set.
- Classical set contains elements that satisfy precise properties of membership while fuzzy set contains elements that satisfy imprecise properties of membership.



Membership Function of Fuzzy set $\tilde{A}$



Membership Function of classical set A

Mathematical Concept

A fuzzy set $\tilde{A}$ in the universe of information U can be defined as a set of ordered pairs and it can be represented mathematically as ·

$$\tilde{A} = \left\{ \left(y, \mu_{\tilde{A}}(y) \right) | y \in U \right\}$$

Here $\mu_{\tilde{A}}(y)$ = degree of membership of y in \widetilde{A}, assumes values in the range from 0 to 1, i.e., $\mu_{\tilde{A}}(y) \in [0,1]$.

Representation of fuzzy set

Let us now consider two cases of universe of information and understand how a fuzzy set can be represented.

Case 1

When universe of information U is discrete and finite −

$$\widetilde{A} = \left\{ \frac{\mu_{\widetilde{A}}(y_1)}{y_1} + \frac{\mu_{\widetilde{A}}(y_2)}{y_2} + \frac{\mu_{\widetilde{A}}(y_3)}{y_3} + \dots \right\}$$

$$= \left\{ \sum_{i=1}^{n} \frac{\mu_{\widetilde{A}}(y_i)}{y_i} \right\}$$

Case 2

When universe of information U is continuous and infinite −

$$\widetilde{A} = \left\{ \int \frac{\mu_{\widetilde{A}}(y)}{y} \right\}$$

Notations for fuzzy sets

1. Pairs *(element, value)* for discrete fuzzy sets (like in the example with the comfortable house), respectively *(generic element, membership function)* for continuous fuzzy sets: e.g. $(x, \mu_{\tilde{A}}(x))$
2. Solely by stating the membership function (for continuous fuzzy sets)
3. As a "sum" for discrete fuzzy sets, respectively "integral" for continuous fuzzy sets (this notation may create confusions !!):

$$\tilde{A} = \sum_{i=1}^{n} \frac{\mu_{\tilde{A}}(x_i)}{x_i} = \frac{\mu_{\tilde{A}}(x_1)}{x_1} + \frac{\mu_{\tilde{A}}(x_2)}{x_2} + \ldots + \frac{\mu_{\tilde{A}}(x_n)}{x_n}$$

$$\tilde{A} = \int \frac{\mu_{\tilde{A}}(x)}{x}$$

Caution, there are neither sums nor integrals here, these are only notations !!!

Properties of fuzzy sets: normal fuzzy sets

1. Normal fuzzy sets
   - A fuzzy set is called *normal* if $\sup_x \mu_{\tilde{A}}(x) = 1$, where sup is the supremum of a fuzzy set
   - The difference between the maximum and the supremum of a set: the maximum belongs to the set, the supremum may belong or not to that set
   - If a fuzzy set is not normal, it can be normalized by dividing its membership function by the supremum of the set, resulting the normalized fuzzy set:

$$\mu_{\tilde{A}_{norm}}(x) = \frac{\mu_{\tilde{A}}(x)}{\sup_x \mu_{\tilde{A}}(x)}$$

Properties of fuzzy sets: normal fuzzy sets

2. The support of a fuzzy set
   - The *support* of a fuzzy set (denoted *supp*) is the crisp set of all $x \in X$ for which $\mu_{\tilde{A}}(x) > 0$
   - In the example with the comfortable house it is the set $supp(\tilde{A}) = \{1, 2, 3, 4, 5, 6, 7\}$
   - Usually the elements of a fuzzy set having the degree of membership equal to 0 are not listed
3. The *(core)* of a fuzzy set:
   - is the crisp set for which $\mu_{\tilde{A}}(x) = 1$
4. The *(boundary)* of a fuzzy set:
   - is the crisp set for which $0 < \mu_{\tilde{A}}(x) < 1$

Properties of a fuzzy set: α-level sets

5. The α-*level sets* (or α-*cuts*):
   - The α-*level set* (where $\alpha \in [0,1]$) of the fuzzy set $\tilde{A}$ having the membership function $\mu_{\tilde{A}}(x)$ is the crisp set A_α for which $\mu_{\tilde{A}}(x) \geq \alpha$
   - We can define *strong* α *cut* as the crisp set A'_α for which $\mu_{\tilde{A}}(x) > \alpha$
   - In the example with the comfortable house, WHERE $\tilde{A} = \{(1, 0.1), (2, 0.5), (3, 0.8), (4, 1.0), (5, 0.7), (6, 0.2)\}$, the α-cuts of the fuzzy set $\tilde{A}$ are:
     - $A_{0.1} = \{1, 2, 3, 4, 5, 6\} = supp\tilde{A}$ (the support of $\tilde{A}$)
     - $A_{0.2} = \{2, 3, 4, 5, 6\}$
     - $A_{0.5} = \{2, 3, 4, 5\}$
     - $A_{0.7} = \{3, 4, 5\}$
     - $A_{0.8} = \{3, 4\}$
     - $A_{1.0} = \{4\} = \text{core}\tilde{A}$

Properties of a fuzzy set: α-level sets

- It can be proved that for any fuzzy set $\tilde{A}$, it holds:

$$\tilde{A} = \bigcup_{\alpha} \alpha \cdot A_{\alpha}$$

- Which means that, any fuzzy set can be written as the union for all the values of α of the product between α and the α-cuts of the fuzzy set
- This property is very important and it connects the fuzzy and the crisp sets
- It is also very useful for proving different properties of fuzzy sets (some properties are easier to be proved for crisp sets)

Properties of a fuzzy set: α-level sets

- We will illustrate this property on the example with the comfortable house:
  - $\alpha \cdot A_\alpha$ is the fuzzy set in which each element will hace the membership function equal with α.
  - $0.1 \cdot A_{0.1} = \{(1, 0.1), (2, 0.1), (3, 0.1), (4, 0.1), (5, 0.1), (6, 0.1)\}$
  - $0.2 \cdot A_{0.2} = \{(2, 0.2), (3, 0.2), (4, 0.2), (5, 0.2), (6, 0.2)\}$
  - $\ldots$
  - $0.8 \cdot A_{0.8} = \{(3, 0.8), (4, 0.8)\}$
  - $1.0 \cdot A_{1.0} = \{(4, 1.0)\}$
  - The union of two or more fuzzy sets is defined as the maximum between their membership function, hence
  - $0.1 \cdot A_{0.1} \cup 0.2 \cdot A_{0.2} \cup \ldots \cup 0.8 \cdot A_{0.8} \cup 1.0 \cdot A_{1.0} =$ $= \{(1, 0.1), (2, max(0.1, 0.2)), (3, max(0.1, 0.2, \ldots, 0.8)),$ $(4, max(0.1, \ldots, 0.8, 1)), \ldots (6, max(0.1, 0.2)\} = \tilde{A}$

Properties of fuzzy sets: convexity

6. Convexity of a fuzzy set
   - A fuzzy set $\tilde{A} \subset X$ is convex if and only if $\forall x_1, x_2 \in X$ and $\forall \lambda \in [0, 1]$ the following relation takes place: $\mu_{\tilde{A}}(\lambda \cdot x_1 + (1 - \lambda) \cdot x_2) \geq min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$
   - The expression $\lambda \cdot x_1 + (1 - \lambda) \cdot x_2$ describes the segment situated between the points having the abscissa x_1 and x_2
   - The expression $\mu_{\tilde{A}}(\lambda \cdot x_1 + (1 - \lambda) \cdot x_2)$ describes the image of this segment through the function $\mu_{\tilde{A}}(x)$
   - **Equivalently, a fuzzy set $\tilde{A}$ is convex iff all its α-level sets are convex**
   - Which means that, if a fuzzy set is not convex, there exist α-level sets of this fuzzy set that are not convex, i.e., there exist segments $x_1^\alpha x_2^\alpha$ which are "interrupted" (are not continues)

Properties of fuzzy sets: cardinality

7. Cardinality of a fuzzy set
   - Cardinality of a finite fuzzy set $\tilde{A} \subset X$, denoted $|\tilde{A}|$ is defined as:
$$|\tilde{A}| = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)$$
   - For a continuous fuzzy set $\tilde{A} \subset X$, its cardinality is defined:
$$|\tilde{A}| = \int_{x} \mu_{\tilde{A}}(x) dx$$
     if the integral exist

7' Relative cardinality of a fuzzy set
   - Is denoted $||\tilde{A}||$
   - Is defined as $||\tilde{A}|| = \frac{|\tilde{A}|}{|X|}$, if it exists, where X is the universe of discourse for the set $\tilde{A}$

How to chose the membership functions

- Like in other aspects of the fuzzy sets theory, there are no clear "recipes" for choosing the membership functions of the fuzzy sets
- If we want to reduce the computations, we will prefer linear membership functions, i.e., triangles and trapeziums
- There are cases when we prefer non-linear membership functions (trigonometric, Gauss-type, etc):
    - There exist researchers that consider that linear membership functions do not provide the best results for some problems, while non-linear functions perform better
    - Sometimes the problem or the domain might need some types of membership functions
    - If we combine fuzzy sets theory with other methods, e.g., neural networks, it can be necessary to use membership functions that are suitable for these methods.