Daubechie's Wavelets

Dr. Sunil Kumar Singh

Department of Mathematics
Mahatma Gandhi Central University
Motihari-845401, Bihar, India

E-mail: sks_math@yahoo.com

Daubechie's Wavelets I

We shall construct the Daubechie's wavelet with one and two vanishing moment. Let us recall the definition of moments.

Definition (Moments): The integral

$$\int_{\mathbb{R}} t^k \, \psi(t) \, dt$$

is called k^{th} moment of ψ, and if

$$\int_{\mathbb{R}} t^k \, \psi(t) \, dt = 0,$$

or equivalently, $\left[\frac{d^k \hat{\psi}(\xi)}{d\xi^k}\right]_{\xi=0} = \hat{\psi}^{(k)}(0) = 0$, we say that ϕ has its k^{th} moment vanishing. If ϕ has M vanishing moments, then

$$\hat{\psi}(0) = \hat{\psi}^{(1)}(0) = \hat{\psi}^{(2)}(0) = \cdots = \hat{\psi}^{(M-1)}(0) = 0.$$

Example

The Haar wavelet $\psi(t)$ is defined by

$$\psi(t) = \begin{cases} 1, & \text{if } 0 \leq t < \frac{1}{2} \\ -1, & \text{if } \frac{1}{2} \leq t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\int_{\mathbb{R}} \psi(t)\, dt = 0, \quad \text{and} \quad \int_{\mathbb{R}} t\, \psi(t)\, dt = -1/4 \neq 0.$$

Therefore, Haar wavelet has one vanishing moment, i.e., $\hat{\psi}(0) = 0$.

Construction of Daubechie's wavelets with M vanishing moments I

Steps involved in construction of Daubechie's wavelets are as follows

S 1. Fix M, the number of vanishing moments.

S 2. Find the polynomial P by using the formula

$$P(y) = \sum_{k=0}^{M-1} \binom{M+k-1}{k} y^k.$$

S 3. Find $|L(\xi)|^2$ by using the formula

$$|L(\xi)|^2 = P\left[\sin^2(\xi/2)\right],$$

where

$$L(\xi) = \sum_{k=0}^{M-1} b_k\, e^{-ik\xi}, \text{ and } b_k \text{ are real.}$$

Construction of Daubechie's wavelets with M vanishing moments II

S 4. Find $\hat{m}(\xi)$ by using the formula

$$\hat{m}(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^M L(\xi).$$

S 5. Put the value of $\hat{m}(\xi)$ obtained in Step 4 in the left side of the following equation

$$\hat{m}(\xi) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} c_k \, e^{-ik\xi},$$

and find the value of the coefficients c_k's.

Construction of Daubechie's wavelets with M vanishing moments III

S 6. Find the Daubechie's scaling function ${}_M\phi(x)$ by using the formula

$${}_M\phi(x) = \sqrt{2} \sum_{k=-\infty}^{\infty} c_k \, \phi(2x - k).$$

S 7. Find the Daubechie's wavelet ${}_M\psi(x)$ by using the formula

$${}_M\psi(x) = \sqrt{2} \sum_{k=-\infty}^{\infty} d_k \, \phi(2x - k)$$

where $d_k = (-1)^k \, \bar{c}_{-k+1}$ or $d_k = (-1)^{(k-1)} \, \bar{c}_{-k-1}$.

The Daubechie's wavelet with one vanishing moment I

For M=1, it follows from

$$P(y) = \sum_{k=0}^{M-1} \binom{M+k-1}{k} y^k$$

that $P(y) = 1$. Then

$$|L(\xi)|^2 = P\left[\sin^2(\xi/2)\right] = 1. \tag{1}$$

Since, for M=1 , $L(\xi) = b_0$ (real constant) and hence we can choose $L(\xi) = 1$. Thus

$$\hat{m}(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^M L(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right) = \frac{1}{2} + \frac{1}{2}e^{-i\xi}. \tag{2}$$

The Daubechie's wavelet with one vanishing moment II

Compare the equation (2) with

$$\hat{m}(\xi) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} c_k \, e^{-ik\xi},$$

we get

$$\frac{1}{\sqrt{2}} c_0 = \frac{1}{2}, \; \frac{1}{\sqrt{2}} c_1 = \frac{1}{2} \text{ and } \frac{1}{\sqrt{2}} c_k = 0, \; \text{for } k \neq 0, 1,$$

$$\implies c_0 = \frac{1}{\sqrt{2}}, \; c_1 = \frac{1}{\sqrt{2}} \text{ and } c_k = 0, \; \text{for } k \neq 0, 1.$$

Hence, the Daubechie's scaling function ${}_1\phi(x)$ is given by

$$\begin{aligned} {}_1\phi(x) &= \sqrt{2} \sum_{k=-\infty}^{\infty} c_k \, \phi(2x - k) \\ &= \sqrt{2} \, [c_0 \phi(2x) + c_1 \phi(2x - 1)] \\ &= \phi(2x) + \phi(2x - 1), \end{aligned}$$

and the Daubechie's wavelet ${}_1\psi(x)$ is given by

$$\begin{aligned}
{}_1\psi(x) &= \sqrt{2}\sum_{k=-\infty}^{\infty} d_k\,\phi(2x-k) \\
&= \sqrt{2}\sum_{k=-\infty}^{\infty} (-1)^k\,\bar{c}_{-k+1}\,\phi(2x-k) \\
&= \sqrt{2}\,\left[(-1)^0\,\bar{c}_1\,\phi(2x) + (-1)\,\bar{c}_0\,\phi(2x-1)\right] \\
&= \phi(2x) - \phi(2x-1)
\end{aligned}$$

which is the Haar wavelet.

The Daubechie's wavelet with two vanishing moment I

For M=2, it follows from

$$P(y) = \sum_{k=0}^{M-1} \binom{M+k-1}{k} y^k$$

that $P(y) = \sum_{k=0}^{1} \binom{k+1}{k} y^k = 1 + 2y$. Then

$$\begin{aligned} |L(\xi)|^2 = P\left[\sin^2(\xi/2)\right] &= 1 + 2\sin^2 \xi/2 \\ &= 2 - \frac{1}{2}\left(e^{i\xi} + e^{-i\xi}\right). \end{aligned} \tag{3}$$

Since, for M=2 we have

$$L(\xi) = \sum_{k=0}^{2-1} b_k e^{-k\xi} = b_0 + b_1 e^{-i\xi}. \tag{4}$$

The Daubechie's wavelet with two vanishing moment II

Hence from (3) and (4), we get

$$
\begin{aligned}
2 - \frac{1}{2}\left(e^{i\xi} + e^{-i\xi}\right) &= L(\xi)\overline{L(\xi)} \\
&= (b_0 + b_1 e^{-i\xi})(b_0 + b_1 e^{i\xi}) \\
&= b_0^2 + b_1^2 + b_0 b_1 \left(e^{i\xi} + e^{-i\xi}\right).
\end{aligned}
$$

Equating the coefficients we get $b_0^2 + b_1^2 = 2$ and $b_0 b_1 = -\frac{1}{2}$.
One solution of these equations is $b_0 = \frac{1}{2}(1+\sqrt{3})$ and $b_1 = \frac{1}{2}(1-\sqrt{3})$.
Consequently,

$$
\begin{aligned}
\hat{m}(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^M L(\xi) &= \left(\frac{1+e^{-i\xi}}{2}\right)^2 (b_0 + b_1 e^{-i\xi}) \\
&= \left(\frac{1+e^{-i\xi}}{2}\right)^2 \left(\frac{1}{2}(1+\sqrt{3}) + \frac{1}{2}(1-\sqrt{3})e^{-i\xi}\right),
\end{aligned}
$$

The Daubechie's wavelet with two vanishing moment III

that is,

$$\hat{m}(\xi) = \frac{1}{8}\{(1+\sqrt{3}) + (3+\sqrt{3})e^{-i\xi} + (3-\sqrt{3})e^{-2i\xi} + (1-\sqrt{3})e^{-3i\xi}\}.$$

Compare the above equation with

$$\hat{m}(\xi) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} c_k \, e^{-ik\xi}$$

we get,

$$\frac{1}{\sqrt{2}}c_0 = \frac{1}{8}(1+\sqrt{3}),\ \frac{1}{\sqrt{2}}c_1 = \frac{1}{8}(3+\sqrt{3}),$$
$$\frac{1}{\sqrt{2}}c_2 = \frac{1}{8}(3-\sqrt{3}),\ \frac{1}{\sqrt{2}}c_3 = \frac{1}{8}(1-\sqrt{3}),$$
$$\text{and,}\ \frac{1}{\sqrt{2}}c_k = 0,\ \text{for } k \neq 0,1,2,3.$$

The Daubechie's wavelet with two vanishing moment IV

Hence,

$$c_0 = \frac{1}{4\sqrt{2}}(1+\sqrt{3}),$$
$$c_1 = \frac{1}{4\sqrt{2}}(3+\sqrt{3}),$$
$$c_2 = \frac{1}{4\sqrt{2}}(3-\sqrt{3}),$$
$$c_3 = \frac{1}{4\sqrt{2}}(1-\sqrt{3}),$$
$$\text{and } c_k = 0, \ \text{ for } k \neq 0,1,2,3.$$

Therefore, the Daubechie's scaling function ${}_2\phi(x)$ is given by

$$\begin{aligned} {}_2\phi(x) &= \sqrt{2}\sum_{k=-\infty}^{\infty} c_k\,\phi(2x-k) \\ &= \sqrt{2}\left\{c_0\,\phi(2x) + c_1\,\phi(2x-1) + c_2\,\phi(2x-2) + c_3\,\phi(2x-3)\right\}. \end{aligned}$$

and the Daubechie's wavelet ${}_2\psi(x)$ is given by

$$\begin{aligned}
{}_2\psi(x) &= \sqrt{2}\sum_{k=-\infty}^{\infty} d_k\,\phi(2x-k) \\
&= \sqrt{2}\sum_{k=-\infty}^{\infty} (-1)^k\,\bar{c}_{-k+1}\,\phi(2x-k) \\
&= \sqrt{2}\,\{c_3\phi(2x+2) - c_2\,\phi(2x+1) + c_1\,\phi(2x) - c_0\phi(2x-1)\}
\end{aligned}$$

It is often referred to as the Daubechies D_4 wavelet since it is generated by four coefficients.

References I

[1] A. Boggess, and F. J. Narcowich, A First Course in Wavelets with Fourier Analysis, Wiley; 2nd ed., 2009.

[2] C. K. Chui : *An Introduction to Wavelets*, Academic Press, New York, 1992.

[3] I. Daubechies : *Ten Lectures on Wavelets*, CBMSNSF Regional Conference Series in Applied Mathematics (SIAM,1992).

[4] Lokenath Debnath and Firdous Ahmad Shah : *Wavelet Transforms and Their Applications*, 2nd ed., Brkhäuser, Boston, 2015.

[5] E. Hernáddez and G. Weiss : *A First Course on Wavelets*, CRC Press, Boca Raton, Florida, 1996.

[6] T. H. Koornwinder : *Wavelets*, World Scientific Publishing Co.Pvt.Ltd., Singapore , 1993.

[7] R. S. Pathak : *The Wavelet transform*, Atlantis Press/ world Scientific, France, 2009.

References II

[8] K. P. Soman, K. I . Ramachandran and N. G. Resmi : *Insight into Wavelets from theory to practice*, Prentice Hall India Learning Private Limited; 3rd edition, 2010.

[9] P. Wojtaszczyk : *A Mathematical Introduction to Wavelets*, Cambridge University Press, 1997.

[10] D. F Walnut : *An Introduction to Wavelet Analysis*, Brkhäuser, 2004.

[11] A. H. Zemanian : *Generalized Integral Transformations*, Interscience Publishers, New York, 1996.

THANK YOU