Solution of Volterra Integral Equation of Second kind by Successive Substitutions

Dr. Rajesh Prasad

Assistant Professor

Department of Mathematics

Mahatma Gandhi Central University

Motihari-845401, Bihar, India

E-mail: rajesh.mukho@gmail.com

rajeshprasad@mgcub.ac.in

April 28, 2020

Let $y(x) = f(x) + \lambda \int_a^x K(x,t)y(t)dt$ (1)

be given Volterra integral equation of the second kind .

Suppose that

(i) Kernel $K(x,t) \neq 0$, is real and continuous in the rectangle R for which $a \leq x \leq b, a \leq t \leq b$.

Also, let $|K(x,t)| \leq M$, in R (2)

(ii) The function $f(x) \neq 0$, is real and continuous in the interval I, for which $a \leq x \leq b$.

Also, let $|f(x)| \leq N$, in I (3)

(iii) The λ is a constant (4)

Therefore, the equation (1) has a unique solution in I and above equation (1) is given by the absolutely and uniformly convergent series

$$y(x) = f(x) + \lambda \int_a^x K(x,t) f(t) dt + \lambda^2 \int_a^x K(x,t) \int_a^t K(t,t_1) f(t_1) dt_1 dt + \cdots \quad (5)$$

Re-writing equation(1), we have

$$y(x) = f(x) + \lambda \int_a^x K(x,t_1) y(t_1) dt_1 \quad (6)$$

Replacing x by t in equation (6), we have

$$y(t) = f(t) + \lambda \int_a^x K(t,t_1) y(t_1) dt_1 \quad (7)$$

Putting the above value of $y(t)$ in equation (1), we obtain

$$y(x) = f(x) + \lambda \int_a^x K(x,t) \left[f(t) + \lambda \int_a^x K(t,t_1) y(t_1) dt_1 \right] dt \quad (8)$$

Re-writing equation (7), we have

$$y(t) = f(t) + \lambda \int_a^x K(t, t_2) y(t_2) dt_2 \tag{9}$$

Replacing t by t_1 in equation (9), we have

$$y(t_1) = f(t_1) + \lambda \int_a^x K(t_1, t_2) y(t_2) dt_2 \tag{10}$$

Substituting the above value of $y(t_1)$ in equation (8), we get

$$y(x) = f(x) + \lambda \int_a^x K(x,t) f(t) dt + \lambda^2 \int_a^x K(x,t) \int_a^t K(t, t_1) f(t_1) dt_1 dt$$

$$+\lambda^3 \int_a^x K(x,t) \int_a^t K(t, t_1) \int_a^{t_1} K(t_1, t_2) y(t_2) dt_2 dt_1 dt \ldots \tag{11}$$

Proceeding the same as above, we have

$$y(x) = f(x) + \lambda \int_a^x K(x,t) f(t) dt + \lambda^2 \int_a^x K(x,t) \int_a^t K(t,t_1) f(t_1) dt_1 dt$$

$$+\lambda^n \int_a^x K(x,t) \int_a^t K(t,t_1) \dots \int_a^{t_{n-2}} K(t_{n-2},t_{n-1}) f(t_{n-1}) dt_{n-1} \dots dt_1 dt$$

$$+R_{n+1}(x), \qquad (12)$$

where

$$R_{n+1}(x) = \lambda^{n+1} \int_a^x K(x,t) \int_a^t K(t,t_1) \dots \int_a^{t_{n-1}} K(t_{n-1},t_n) y(t_n) dt_n \dots dt_1 dt$$

$$, \qquad (13)$$

Now, let us consider the infinite series

$$f(x) + \lambda \int_a^x K(x,t) f(t)dt + \lambda^2 \int_a^x K(x,t) \int_a^t K(t,t_1) f(t_1) dt_1 dt + \cdots \qquad (14)$$

In view of the assumptions (i) and (ii), each term of the series equation (14) is continuous in interval I. It follows that the series equation (14) is continuous in I, then its converges uniformly in I.

Let

$$V_n(x) = \lambda^n \int_a^x K(x,t) \int_a^t K(t,t_1) \ldots \int_a^{t_{n-2}} K(t_{n-2},t_{n-1}) f(t_{n-1}) dt_{n-1} \ldots dt_1 dt \qquad (15)$$

From equation (15), we have

$$|V_n(x)| \le |\lambda^n| N M^n \frac{(x-a)^n}{n!}$$ Using eq.(2) and (3)

$$|V_n(x)| \le |\lambda^n| N M^n \frac{(b-a)^n}{n!}, \qquad a \le x \le b$$

$$|V_n(x)| \le |\lambda^n| \frac{N[M(b-a)]^n}{n!}, \qquad a \le x \le b \qquad (16)$$

Clearly, the series for which the positive constant

$|\lambda^n| \frac{N[M(b-a)]^n}{n!}$ is the general expression for the nth term,

is convergent for all values of $\lambda, N, M, (b-a)$.

Therefore, from equation (16), it follows that the series equation (14) converges absolutely and uniformly. If eq.(1) has a continuous solution, it must be expressed by eq.(12). If $y(x)$ is continuous in I, $|y(x)|$ must have a maximum value Y. Therefore $|y(x)| \leq Y$ (17)

Now, from equation (13), we have

$$|R_{n+1}(x)| = \left| \lambda^{n+1} \int_a^x K(x,t) \int_a^t K(t,t_1) \ldots \int_a^{t_{n-1}} K(t_{n-1},t_n) y(t_n) dt_n \ldots dt_1 dt \right|$$

$$|R_{n+1}(x)| \leq \frac{|\lambda|^{n+1} Y M^{n+1} (x-a)^{n+1}}{(n+1)!}$$

$$|R_{n+1}(x)| \leq \frac{|\lambda|^{n+1} Y M^{n+1} (b-a)^{n+1}}{(n+1)!}, (a \leq x \leq b)$$

Hence $\lim_{n\to\infty} R_{n+1}(x) = 0$.

It follows that the function $y(x)$ satisfying equation (12) is the continuous function given by the series eq. (14).

Q. Determine the resolvent kernels for the Fredholm integral equation having kernels:

(i) $K(x,t) = e^{x+t},\ a = 0, b = 1$

(ii) $K(x,t) = (1+x)(1-t),\ a = -1, b = 1$

Sol: We know that iterated kernels $K_m(x,t)$

$$K_1(x,t) = K(x,t) \quad (1)$$

$$K_m(x,t) = \int_0^1 K(x,z)K_{m-1}(z,t)dz \quad (2)$$

From equation (1) $K_1(x,t) = K(x,t) = e^{x+t}$ (3)

Putting $n = 2$ in equation (2), we have

$$K_2(x,t) = \int_0^1 K(x,z)K_1(z,t)dz$$

$$K_2(x,t) = e^{x+t}\left(\frac{e^2-1}{2}\right) \quad (4)$$

Putting $n = 3$ in equation (2), we have

$$K_3(x,t) = \int_0^1 K(x,z)K_2(z,t)dz$$

$$K_3(x,t) = e^{x+t}\left(\frac{e^2-1}{2}\right)^2 \qquad (5)$$

as before ….. …..and so on. Now, observing the equation (3), (4) and (5), we may write

$$K_m(x,t) = e^{x+t}\left(\frac{e^2-1}{2}\right)^{m-1}, m = 1,2,3 \dots \dots \qquad (6)$$

Now, the required resolvent kernel is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) = \sum_{m=1}^{\infty} \lambda^{m-1} e^{x+t} \left(\frac{e^2-1}{2}\right)^{m-1}$$

$$R(x,t;\lambda) = e^{x+t} \sum_{m=1}^{\infty} \left[\lambda\left(\frac{e^2-1}{2}\right)\right]^{m-1} \qquad (7)$$

But $\sum_{m=1}^{\infty} \left[\lambda\left(\frac{e^2-1}{2}\right)\right]^{m-1} = 1 + \lambda\left(\frac{e^2-1}{2}\right) + \left\{\lambda\left(\frac{e^2-1}{2}\right)\right\}^2 + \cdots$

Which is an infinite geometric series with common ratio $\lambda\left(\frac{e^2-1}{2}\right)$

Therefore, $$\sum_{m=1}^{\infty} \left[\lambda\left(\frac{e^2-1}{2}\right)\right]^{m-1} = \frac{2}{(2-\lambda(e^2-1))} \qquad (8)$$

Provided $\left|\lambda\left(\frac{e^2-1}{2}\right)\right| < 1$ (9)

Now, using equation (8) and (9), equation (7) reduces to

$$R(x,t;\lambda) = \frac{2e^{x+t}}{\left(2-\lambda(e^2-1)\right)}$$ Provided $|\lambda| < \frac{2}{e^2-1}$

Question: Solve the following integral equation by the method of successive approximations

(1) $$y(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xty(t)dt$$

(2) $$y(x) = x + \lambda \int_0^1 xty(t)dt$$

(3) $$y(x) = \sin x - \frac{x}{4} + \frac{1}{4} \int_0^1 xty(t)dt$$

(4) $$y(x) = \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{2} + \frac{1}{2} \int_0^1 ty(t)dt$$

Try to yourself above given problem

References:

- M.R. Spiegel, Fourier Analysis with Applications to Boundary Value Problems,Schaum's Series, Tata McGraw-Hill.
- Francis B. Hildebrand, Methods of Applied Mathematics, Dover Publications.
- M.D. Raisinghania, Integral Equations, S. Chand and Company.
- R.P. Kanwall, Linear Integral Equations. Theory and Techniques. Academic press, New York.
- A.S. Gupta, Calculus of Variation with Applications, Prentice Hall of India.
- Naveen Kumar, An Elementary Course of Variational Problems in Calculus, Narosa Publishing House, New Delhi.

Thank you