Solution of VIE and FIE of Second kinds

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April 28, 2020

(1)

Th The m^{th} iterated kernel $K_m(x,t)$ satisfies the relation

$$K_m(x,t)=\int_a^b K_r(x,y)\,K_{m-r}(y,t)\,dy$$

where r is any positive integer less than m.

Proof: Let $K_1(x,t)=K(x,t)$ — (1)

then m^{th} iterated kernel $K_m(x,t)$ is defined as

$$K_m(x,t)=\int_a^b K(x,s)\,K_{m-1}(s,t)\,ds,\quad m=2,3\ldots$$ — (2)

Rewriting above eqn (2), we have

$$K_m(x,t)=\int_a^b K(x,s_1)\,K_{m-1}(s_1,t)\,ds_1$$ — (3)

$m \longrightarrow m-1$ in eqn (2),

we ha... b

have
$$K_m(x,t) = \int_a^b K(x,s_1) K_{m-1}(s_1,t)\, ds_1 \quad \text{--- (3)}$$

$m \longrightarrow m-1$ in eqn (2),

we have
$$K_{m-1}(x,t) = \int_a^b K_1(x,s) K_{m-2}(s,t)\, ds$$

$$= \int_a^b K(x,s_2) K_{m-2}(s_2,t)\, ds_2$$

$$K_{m-1}(s_1,t) = \int_a^b K_1(s_1,s_2) K_{m-2}(s_2,t)\, ds_2 \quad \text{--- (4)}$$

Using eqn (4), then eqn (3) reduces to

$$K_m(x,t) = \int_a^b K(x,s_1) \left[\int_a^b K(s_1,s_2) K_{m-2}(s_2,t)\, ds_2 \right] ds_1$$

$$= \int_a^b \int_a^b K(x,s_1) K(s_1,s_2) K_{m-2}(s_2,t)\, ds_2\, ds_1$$

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(2)

Similarly, we obtain

$$K_m(x,t) = \int_a^b \int_a^b \cdots \int_a^b K(x,s_1)K(s_1,s_2)K(s_2,s_3) \cdots$$

$$\cdots K_1(s_{m-1},t)\, ds_{m-1} \cdots ds_2\, ds_1$$

$$K_m(x,t) = \int_a^b \int_a^b \cdots \int_a^b K(x,s_1)K(s_1,s_2) \cdots\cdots$$

$$\cdots K(s_{r-1},s_r)K(s_r,s_{r+1}) \cdots K(s_{m-1},t)\, ds_{m-1} \cdots ds_2\, ds_1$$

$$\text{—— (5)}$$

Similarly, the above eqn (5), we may also write

$$K_r(x,y) = \underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{(r-1)\text{th order integral}} K(x,u_1)\, K(u_1,u_2) \cdots K(u_{r-1},y)\, du_{r-1} \cdots du_2\, du_1$$

$$\text{— (6)}$$

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$$K_{r}(x,y) = \underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{(r-1)\text{th order integral}} K(x,u_1)\,K(u_1,u_2)\cdots K(u_{r-1},y)\,du_{r-1}\cdots du_2\,du_1 \quad \text{--- (6)}$$

$$K_{m-r}(y,t) = \int_a^b \int_a^b \cdots \int_a^b K(y,v_1)\,K(v_1,v_2)\cdots K(v_{m-r-1},t)\,dv_{m-r-1}\cdots dv_2\,dv_1 \quad \text{--- (7)}$$

Now, $\int_a^b K_{r}(x,y)\,K_{m-r}(y,t)\,dy$

$$= \int_a^b \Big[\Big\{ \int_a^b \int_a^b \cdots \int_a^b K(x,u_1)\,K(u_1,u_2)\cdots K(u_{r-1},y)\,du_{r-1}\cdots du_2\,du_1 \Big\}$$

$$\times \Big\{ \int_a^b \int_a^b \cdots \int_a^b K(y,v_1)\,K(v_1,v_2)\cdots K(v_{m-r-1},t)\,dv_{m-r-1}\cdots dv_2\,dv_1 \Big\} \Big]$$

$$= \int_a^b \int_a^b \cdots \int_a^b K(x,u_1)\,K(u_1,u_2)\cdots K(u_{r-1},y) \times K(y,v_1)$$

$$K(v_1,v_2)\cdots K(v_{m-r-1},t)\,dv_{m-r-1}\cdots dv_2\,dv_1\,dy\,du_{r-1}\cdots du_2 \cdots du_2\,du_1$$

[Interchanging the Order of Integration] --- (8)

The Order of the multiple integral on RHS of eqn (8) is $1 + r - 1 + m - r - 1 = m - 1$

We have already prove that the order of the multiple integral on RHS of eqn (5) is also $m-1$.

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Therefore, multiple integrals involved in $K_m(x,t)$ and

$\int_a^b K_n(x,y)K_{m-n}(y,t)\,dy$ are both of the same order, $(m-1)^{th}$

$$K_m(x,t) = \int_a^b K_n(x,y)\,K_{m-n}(y,t)\,dy$$

changing the variables of integrations

$$\begin{array}{cccccccccc} u_1 & u_2 & \cdots & u_{n-1} & y & v_1 & v_2 & \cdots & v_{m-n-1} \\ \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\ s_1 & s_2 & & s_{n-1} & s_n & s_{n+1} & s_{n+2} & & s_{m-1} \end{array}$$

Solution of Fredholm Integral Equation of the second kind:—

Fredholm integral eqn of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\,y(t)\,dt \quad \text{—(1)}$$

(i) Suppose that kernel $K(x,t) \neq 0$ is

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Fredholm integral eq^n of the Second kind

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\, y(t)\, dt \quad \text{--- (1)}$$

(i) Suppose that kernel $K(x,t) \not\equiv 0$ is real and continuous in the rectangle R, for which $a \le x \le b$, $a \le t \le b$

also let $|K(x,t)| \le M$, in R — (2)

(ii) Let $f(x) \not\equiv 0$ is real and continuous in the ~~[illegible]~~ interval I, for which $a \le x \le b$

$|f(x)| \le N$, in I — (3)

(iii) λ is a constant s.t. $|\lambda| < \frac{1}{M(b-a)}$ — (4)

Then eq^n (1) has a unique continuous solution in I and the solution is given by absolutely and uniformly convergent series

$$y(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) f(t_1)\, dt_1\, dt + \cdots \quad (5)$$

Now, re-writing eqn (1)

$$y(x) = f(x) + \lambda \int_a^b K(x, t_1)\, y(t_1)\, dt_1 \qquad (6)$$

Replacing x by t in above eqn (6), we get

$$y(t) = f(t) + \lambda \int_a^b K(t, t_1)\, y(t_1)\, dt_1 \qquad (7)$$

Substituting the value of $y(t)$ in eqn (1), we obtain

$$y(x) = f(x) + \lambda \int_a^b K(x, t) \left[f(t) + \lambda \int_a^b K(t, t_1)\, y(t_1)\, dt_1 \right] dt$$

$$y(x) = f(x) + \lambda \int_a^b K(x, t) f(t)\, dt + \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1)\, y(t_1)\, dt_1\, dt \qquad (8)$$

Re-writing eqn (7), we have

$$y(t) = f(t) + \lambda \int_a^b K(t, t_2)\, y(t_2)\, dt_2 \qquad (9)$$

$$y(x)=f(x)+\lambda\int_a^b K(x,t)f(t)\,dt+\lambda^2\int_a^b K(x,t)\int_a^b K(t,t_1)y(t_1)\,dt_1\,dt \quad \text{--- (8)}$$

Re-writing eqn (7), we have

$$y(t)=f(t)+\lambda\int_a^b K(t,t_2)y(t_2)\,dt_2 \quad \text{--- (9)}$$

In above eqn (9) replacing t by t_1, then we get

$$y(t_1)=f(t_1)+\lambda\int_a^b K(t_1,t_2)y(t_2)\,dt_2 \quad \text{--- (10)}$$

Putting the above value of $y(t_1)$ in eqn (8), we get

$$y(x)=f(x)+\lambda\int_a^b K(x,t)f(t)\,dt+\lambda^2\int_a^b K(x,t)\int_a^b K(t,t_1)\left[f(t_1)+\lambda\int_a^b K(t_1,t_2)y(t_2)\,dt_2\right]dt_1\,dt$$

$$y(x)=f(x)+\lambda\int_a^b K(x,t)f(t)\,dt+\lambda^2\int_a^b K(x,t)\int_a^b K(t,t_1)f(t_1)\,dt_1\,dt+\lambda^3\int_a^b K(x,t)\int_a^b K(t,t_1)\int_a^b K(t_1,t_2)\,y(t_2)\,dt_2\,dt_1\,dt \quad \text{--- (11)}$$

Similarly, we have

$$y(x) = f(x) + \lambda\int_a^b K(x,t)f(t)\,dt + \lambda^2\int_a^b K(x,t)\int_a^b K(t,t_1) f(t_1)\,dt_1\,dt + \dots + \lambda^n\int_a^b K(x,t)\int_a^b K(t,t_1)\cdots\int_a^b K(t_{n-2},t_{n-1}) f(t_{n-1})\,dt_{n-1}\cdots dt_1\,dt + R_{n+1}(x) \quad (12)$$

where

$$R_{n+1}(x) = \lambda^{n+1}\int_a^b K(x,t)\int_a^b K(t,t_1)\cdots\int_a^b K(t_{n-1},t_n)\,y(t_n)\,dt_n\cdots dt_1\,dt \quad (13)$$

Now, let us consider the infinite series

$$f(x) + \lambda\int_a^b K(x,t)f(t)\,dt + \lambda^2\int_a^b K(x,t)\int_a^b K(t,t_1) f(t_1)\,dt_1\,dt + \dots \quad (14)$$

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$$f(x)+\lambda\int_a^b K(x,t)f(t)\,dt+\lambda^2\int_a^b K(x,t)\int_a^b K(t,t_1)f(t_1)\,dt_1\,dt$$
$$+\cdots\cdots \quad -(14)$$

In view of the assumptions (i) & (ii), each term of the series eqn (14) is continuous in I. It follows that the series eqn (14) is also continuous in I, then it converges uniformly in I.

Let $V_n(x)$ be the general term of series eqn (14),

$$V_n(x)=\lambda^n\int_a^b K(x,t)\int_a^b K(t,t_1)\cdots\int_a^b K(t_{n-2},t_{n-1})$$
$$f(t_{n-1})\,dt_{n-1}\cdots dt_1\,dt \quad -(15)$$

From eqn (15), we have

$$|V_n(x)|=\left|\lambda^n\int_a^b K(x,t)\int_a^b K(t,t_1)\cdots\int_a^b K(t_{n-2},t_{n-1})\right.$$
$$\left.f(t_{n-1})\,dt_{n-1}\cdots dt_1\,dt\right|$$

$$|V_n(x)|\le|\lambda|^n NM^n(b-a)^n, \text{ by eq}^n (2) \,\&\, (3)$$
$$-(16)$$

The series of which is a general term converges only when

$$|\lambda| M(b-a) < 1$$

$$|\lambda| < 1/M(b-a)$$

which holds in view of assumption (iii)

It follows that the series eqn (14) converges absolutely and uniformly when eqn (4) holds.

If $y(x)$ is continuous in I, $|y(x)|$ must have a maximum value $y(x)$;

$$|y(x)| \leq Y \quad \text{———} \quad (17)$$

Now, from eqn (13) we have

$$|R_{n+1}(x)| = \left| \lambda^{n+1} \int_a^b k(x,t) \int_a^b k(t,t_1) \cdots \int_a^b k(t_{n-1},t_n) \right.$$

$$\left. y(t_n)\, dt_n \ldots dt_1\, dt \right|$$

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assumption (iii)

It follows that the series eqn (14) converges absolutely and uniformly when eqn (4) holds.

If $y(x)$ is continuous in I, $|y(x)|$ must have a maximum value Y:

$$|y(x)| \le Y \quad \text{--- (17)}$$

Now, from eqn (13) we have

$$|R_{n+1}(x)| = \left|\lambda^{n+1}\int_a^b K(x,t)\int_a^b K(t,t_1)\cdots\int_a^b K(t_{n-1},t_n)\, y(t_n)\, dt_n \cdots dt_1\, dt\right|$$

$$\therefore |R_{n+1}(x)| \le |\lambda|^{n+1} Y M^{n+1} (b-a)^{n+1},$$

using eqn (2) & (17)

$\because$ eqn (4) holds,

then $\lim_{n \to \infty} R_{n+1}(x) = 0$

It follows that the function $y(x)$ satisfying eqn (12) is the continuous function given by the series eqn (14).

Solution of Volterra integral equation of the second kind

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Th!. let $R(x,t,\lambda)$ be the resolvent kernel of a Fredholm integral eqn

$$y(x) = f(x) + \lambda \int_a^b k(x,t)\, y(t)\, dt,$$

then the resolvent kernel satisfies the integral eqn

$$R(x,t,\lambda) = k(x,t) + \lambda \int_a^b k(x,z) R(z,t,\lambda) d$$

Proof!. we know that the resolvent kernel is given by

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \quad \text{--- (1)}$$

let $K_1(x,t) = K(x,t)$ --- (2)

and $$K_m(x,t) = \int_a^b k(x,z) K_{m-1}(z,t)\, dz \quad \text{--- (3)}$$

Now, from eqn (1), we have

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Proof:- We know that the resolvent kernel is given by $R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$. —(1)

let $K_1(x,t) = K(x,t)$ —(2)

and $K_m(x,t) = \int_a^b K(x,z) K_{m-1}(z,t)\, dz$ —(3)

Now, from eq^n (1), we have

$$R(x,t,\lambda) = K_1(x,t) + \sum_{m=2}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$= K(x,t) + \sum_{m=2}^{\infty} \lambda^{m-1} \int_a^b K(x,z) K_{m-1}(z,t)\, dz$$

using eq^n (2) & (3).

putting $m-1 = n$ in above expression, we have

$$R(x,t,\lambda) = K(x,t) + \sum_{n=1}^{\infty} \lambda^{n} \int_a^b K(x,z) K_n(z,t)\, dz$$

$n \to M$

$$R(x,t,\lambda) = K(x,t) + \sum_{M=1}^{\infty} \lambda^{M} \int_a^b K(x,z) K_M(z,t)\, dz$$

(8)

$$R(x,t,\lambda) = k(x,t) + \lambda \sum_{M=1}^{\infty} \lambda^{M-1} \int_a^b k(x,z) k_M(z,t)\, dz$$

$$= k(x,t) + \lambda \sum_{M=1}^{\infty} \int_a^b \lambda^{M-1} k_M(z,t) k(x,z)\, dz$$

$$= k(x,t) + \lambda \int_a^b \left[\sum_{M=1}^{\infty} \lambda^{M-1} k_M(z,t) \right] k(x,z)\, dz$$

$$= k(x,t) + \lambda \int_a^b R(z,t,\lambda) k(x,z)\, dz$$

using eqn (1)

Therefore, the required soln

$$R(x,t,\lambda) = k(x,t) + \lambda \int_a^b k(x,z) R(z,t,\lambda)\, dz$$

Try to yourself

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$$R(x,t,\lambda) = K(x,t) + \lambda \int_a^b K(x,z) R(z,t,\lambda) dz$$

Try to yourself

\#

The series for the resolvent kernel $R(x,t,\lambda)$

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

is absolutely and uniformly convergent for all values of x and t in the circle $|\lambda| < B^{-1}$.

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Q. Find the iterated kernels or functions for the following kernels (Fredholm Integral eqn)

(i) $K(x,t) = x + \sin t$, $a = -\pi$ & $b = \pi$

(ii) $K(x,t) = e^{x} \cos t$, $a = 0$ & $b = \pi$

Sol^n (i) :- Given that

$$K(x,t) = x + \sin t$$

Let $K_1(x,t) = K(x,t)$ and —— (1)

$$K_m(x,t) = \int_{-\pi}^{\pi} K(x,z) K_{m-1}(z,t)\, dz \quad \text{—— (2)}$$

putting $m = 2$ in above eqn (2), we obtain

$$K_2(x,t) = \int_{-\pi}^{\pi} K(x,z) K_1(z,t)\, dz$$

$$= \int_{-\pi}^{\pi} (x + \sin z)(z + \sin t)\, dz$$

$$K_m(x,t) = \int_{-\pi}^{\pi} K(x,z) K_{m-1}(z,t)\, dz \quad \text{—} (2)$$

putting $m=2$ in above eqn (2), we obtain

$$K_2(x,t) = \int_{-\pi}^{\pi} K(x,z) K_1(z,t)\, dz$$

$$= \int_{-\pi}^{\pi} (x+\sin z)(z+\sin t)\, dz$$

$$= x\int_{-\pi}^{\pi} z\, dz + \sin t \int_{-\pi}^{\pi} \sin z\, dz + x \sin t \int_{-\pi}^{\pi} dz + \int_{-\pi}^{\pi} z \sin z\, dz$$

$$K_2(x,t) = 2\pi x \sin t + 2\pi$$

$$K_2(x,t) = 2\pi(1 + x\sin t) \quad \text{—} (3)$$

Next, putting $m=3$, in eqn (2), we have

$$K_3(x,t) = \int_{-\pi}^{\pi} K(x,z) K_2(z,t)\, dz$$

$$= \int_{-\pi}^{\pi} (x+\sin z)\, 2\pi (1 + z\sin t)\, dz$$

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(16)

$$K_3(x,t) = 2\pi \int_{-\pi}^{\pi} (x + xz\sin t + \sin z + z\sin z \sin t)\, dz$$

$$= 4\pi^2 x + 4\pi^2 \sin t$$

$$K_3(x,t) = 4\pi^2 (x + \sin t)$$

$$K_3(x,t) = 4\pi^2 K_1(x,t) \quad \text{— (4)}$$

Now, putting $n = 4$, we have

$$K_4(x,t) = \int_{-\pi}^{\pi} K(x,z) K_3(z,t)\, dz$$

$$K_4(x,t) = \int_{-\pi}^{\pi} (x + \sin z)\, 4\pi^2 (z + \sin t)\, dz$$

$$K_4(x,t) = 4\pi^2 K_2(x,t)$$

Similarly, we have — (5)

$$K_5(x,t) = 4\pi^2 K_3(x,t) = 16\pi^4 K_1(x,t)$$

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$$K_4(x,t) = \int_{-\pi}^{\pi} (x + \sin z)\, 4\pi^2 (z + \sin t)\, dz$$

$$K_4(x,t) = 4\pi^2 K_2(x,t)$$

Similarly, we have — (5)

$$K_5(x,t) = 4\pi^2 K_3(x,t) = 16\pi^4 K_1(x,t)$$

$$K_6(x,t) = 4\pi^2 K_4(x,t) = 16\pi^4 K_2(x,t)$$

— (6)

Therefore, the required iterated kernels $K_n(x,t)$ as follows

if $n = 2m-1$, then

$$K_{2m-1}(x,t) = (2\pi)^{2m-2} (x + \sin t), \quad m = 1, 2, 3 \ldots$$

if $n = 2m$, then

$$K_{2m}(x,t) = (2\pi)^{2m-1} (1 + x \sin t), \quad m = 1, 2, 3$$

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(ii) Given that $k_1(x,t) = e^x \cos t$

Let $k_1(x,t) = k(x,t) = e^x \cos t$ — (1)

and $k_n(x,t) = \int_0^{\pi} k(x,z)\, k_{n-1}(z,t)\, dz$ — (2)

putting $n=2$, in above eqn (2), we obtain

$$k_2(x,t) = \int_0^{\pi} k(x,z)\, k_1(z,t)\, dz$$

$$= \int_0^{\pi} e^x \cos z \; e^z \cos t \, dz$$

$$= e^x \cos t \int_0^{\pi} e^z \cos z \, dz$$

$$= e^x \cos t \left[\frac{e^z}{2} (\cos z + \sin z) \right]_0^{\pi}$$

$$= e^x \cos t \times \left[-\left(\tfrac{1}{2}\right) e^{\pi} - \tfrac{1}{2} \right]$$

$$k_2(x,t) = (-1) \frac{(1+e^{\pi})}{2} e^x \cos t \quad \text{— (3)}$$

Next, ~~[illegible]~~ putting $n=3$,

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$$K_2(x,t) = (-1)\frac{(1+e^{\pi})}{2} e^{x}\cos t \quad \text{--- (3)}$$

Next, ~~[illegible]~~ putting $n=3$, in above eqn (2), we get

$$K_3(x,t) = \int_0^{\pi} K(x,z) K_2(z,t)\, dz$$

$$= \int_0^{\pi} e^{x}\cos z \left\{-\frac{(1+e^{\pi})}{2} e^{z}\cos t\right\} dz$$

$$= -\frac{(1+e^{\pi})}{2} e^{x}\cos t \int_0^{\pi} e^{z}\cos z\, dz$$

$$= (-1)^2 \left(\frac{1+e^{\pi}}{2}\right)^2 e^{x}\cos t \quad \text{--- (4)}$$

and so on,

we get, the iterated kernels

$$K_n(x,t) = (-1)^{n-1}\left(\frac{1+e}{2}\right)^{n-1} e^x \cos t$$

$n = 1, 2, 3, \ldots$

Try to yourself

(1) Find the iterated kernels or functions for the following kernels (F.I.E)

(i) $K(x,t) = \sin(x-2t)$, $0 \le x \le 2\pi$

(ii) $K(x,t) = x - t$, $a = 0$, $b = 1$.

(2) Determine the resolvent kernels for the FIE having kernels

(i) $K(x,t) = e^{x+t}$, $a = 0$ & $b = 1$

(ii) $K(x,t) = (1+x)(1-t)$; $a = -1$, & $b = 1$

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