PHYS 3014: Statistical Mechanics

Lecture Notes
Part 16

Bose Gas : Specific Heat

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Specific heat

The internal energy of the Bose gas

$$E = -\left[\frac{\partial}{\partial \beta} \ln Z\right]_{z,V} \quad \text{--- (1)}$$

$$= +\frac{\partial}{\partial \beta} \cdot \sum_{\epsilon} \ln\left(1 - z e^{-\beta\epsilon}\right)$$

$$= \sum_{\epsilon} \frac{z e^{-\beta\epsilon} \cdot \epsilon}{1 - z e^{-\beta\epsilon}}$$

$$= \sum_{\epsilon} \frac{\epsilon}{z^{-1} e^{\beta\epsilon} - 1}$$

For large V, $\sum_{\epsilon} \longrightarrow \int g(\epsilon)\, d\epsilon$

$$E = \frac{2\pi V}{h^3} (2m)^{3/2} \int_0^{\infty} \frac{\epsilon \cdot \epsilon^{1/2}\, d\epsilon}{z^{-1} e^{\beta\epsilon} - 1}$$

$$= \frac{2\pi V}{h^3} \cdot (2m)^{3/2} (KT)^{5/2} \int_0^{\infty} \frac{x^{3/2}\, d\epsilon}{z^{-1} e^{x} - 1} \quad \because x = \beta\epsilon \quad \text{-- (2)}$$

$$E = \frac{2\pi V}{h^3}(2m)^{3/2}(KT)^{5/2}\int_0^\infty \frac{x^{3/2}\,dx}{z^{-1}e^x - 1}$$

$$= \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{2\pi V}{h^3}\cdot(2m)^{3/2}\cdot(KT)^{5/2}\frac{1}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\int_0^\infty \frac{x^{3/2}\,dx}{z^{-1}e^x - 1}$$

$$= \frac{3}{2}KT\,\frac{V(2\pi mKT)^{3/2}}{h^3}\cdot g_{5/2}(z)$$

$$= \frac{3}{2}KT\,\frac{V}{\lambda^3}\,g_{5/2}(z) \quad \text{--- (3)}$$

For $T < T_c$, we have $z = 1$ independent of temperature so the specific heat

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V}$$

$$= \frac{3}{2}KV\,\zeta\left(\frac{5}{2}\right)\left[\frac{\partial}{\partial T}\left(\frac{T}{\lambda^3}\right)\right]_{N,V}$$

$$= \frac{3}{2}KV\,\zeta\left(\frac{5}{2}\right)\cdot\frac{5}{2}\cdot\frac{1}{\lambda^3} \quad \text{--- (4)}$$

$$\therefore \frac{C_v}{NK} = \frac{15}{4} \zeta\left(\frac{5}{2}\right) \frac{V}{N\lambda^3}$$

$$= \frac{15}{4} \times 1.341 \frac{V}{N} \frac{(2\pi m K T)^{3/2}}{h^3} \quad \text{---} (5) \qquad \because \zeta\left(\frac{5}{2}\right) = 1.341$$

we know that $T_c = \frac{h^2}{2\pi m K} \left[\frac{N}{V\zeta\left(\frac{3}{2}\right)}\right]^{2/3}$ --- (6)

$$\therefore T_c^{3/2} = \frac{h^3}{(2\pi m K)^{3/2}} \frac{N}{V\zeta\left(\frac{3}{2}\right)}, \quad \zeta\left(\frac{3}{2}\right) = 2.612$$

$$\therefore \frac{C_v}{NK} = \frac{15}{4} \times 1.341 \times \frac{1}{\zeta\left(\frac{3}{2}\right)} \cdot \frac{T^{3/2}}{T_c^{3/2}}$$

$$= \frac{15}{4} \times 1.341 \times \frac{1}{2.612} \left(\frac{T}{T_c}\right)^{3/2}$$

$$\therefore \boxed{\frac{C_v}{NK} = 1.925 \left(\frac{T}{T_c}\right)^{3/2}}$$ In the condensed state. --- (7)

At $T = T_c$, $\frac{C_v}{NK} = 1.925$

For $T > T_c$, z will be function of T. but $N_0 \simeq 0$, so we have $N = \frac{V}{\lambda^3} g_{3/2}(z)$ and $z << 1$. classical limit

and $E = \frac{3}{2} KT \frac{V}{\lambda^3} g_{5/2}(z)$

$$\Rightarrow \quad E = \frac{3}{2} NKT \frac{g_{5/2}(z)}{g_{3/2}(z)} \quad \text{--- (8)}$$

$\therefore$ specific heat

$$C_V = \frac{3}{2} NK \left[\frac{\partial}{\partial T} \left\{ T \frac{g_{5/2}(z)}{g_{3/2}(z)} \right\} \right]_{N,V}$$

$$\therefore \quad \frac{C_V}{NK} = \frac{3}{2} \frac{g_{5/2}(z)}{g_{3/2}(z)} + \frac{3}{2} T \left[\frac{\partial}{\partial T} \left(\frac{g_{5/2}(z)}{g_{3/2}(z)} \right) \right]_{N,V} \quad \text{--- (9)}$$

The recursion relation for $g_\nu(z)$ is

$$\frac{\partial}{\partial z} g_\nu(z) = \frac{1}{z} \cdot g_{\nu-1}(z)$$

$$\therefore \quad \frac{\partial}{\partial T}\left[\frac{g_{5/2}(z)}{g_{3/2}(z)}\right] = \frac{\partial z}{\partial T} \cdot \frac{1}{z}\left[\frac{g_{3/2}(z) \cdot g_{3/2}(z) - g_{5/2}(z)\, g_{1/2}(z)}{[g_{3/2}(z)]^2}\right] \quad \text{--- (10)}$$

and $\dfrac{\partial}{\partial T}\left[g_{3/2}(z)\right] = \dfrac{\partial}{\partial T}\left(\dfrac{N\lambda^3}{V}\right) = -\dfrac{3}{2T}\left(\dfrac{N\lambda^3}{V}\right) = -\dfrac{3}{2T}\, g_{3/2}(z)$

and $\dfrac{\partial}{\partial T}\left[g_{3/2}(z)\right] = \dfrac{\partial z}{\partial T} \cdot \dfrac{\partial}{\partial z}\left[g_{3/2}(z)\right] = \dfrac{\partial z}{\partial T} \cdot \dfrac{1}{z}\, g_{1/2}(z)$

$$\Rightarrow \quad \frac{\partial z}{\partial T} \cdot \frac{1}{z}\, g_{1/2}(z) = -\frac{3}{2T}\, g_{3/2}(z)$$

or, $$\frac{\partial z}{\partial T} = -\frac{3z}{2T}\, \frac{g_{3/2}(z)}{g_{1/2}(z)} \quad \text{--- (11)}$$

Using (10) and (11)

$$\frac{\partial}{\partial T}\left[\frac{g_{5/2}(z)}{g_{3/2}(z)}\right] = -\frac{3z}{2T} \cdot \frac{1}{z} \cdot \frac{g_{3/2}(z)}{g_{1/2}(z)}\left[1 - \frac{g_{5/2}(z)\, g_{1/2}(z)}{(g_{3/2}(z))^2}\right]$$

$$= -\frac{3}{2T}\left[\frac{g_{3/2}(z)}{g_{1/2}(z)} - \frac{g_{5/2}(z)}{g_{3/2}(z)}\right] \quad \text{---} (12)$$

So equation (9) becomes

$$\frac{C_v}{NK} = \frac{3}{2}\frac{g_{5/2}(z)}{g_{3/2}(z)} + \frac{3}{2}T\cdot\left(-\frac{3}{2T}\right)\left[\frac{g_{3/2}(z)}{g_{1/2}(z)} - \frac{g_{5/2}(z)}{g_{3/2}(z)}\right]$$

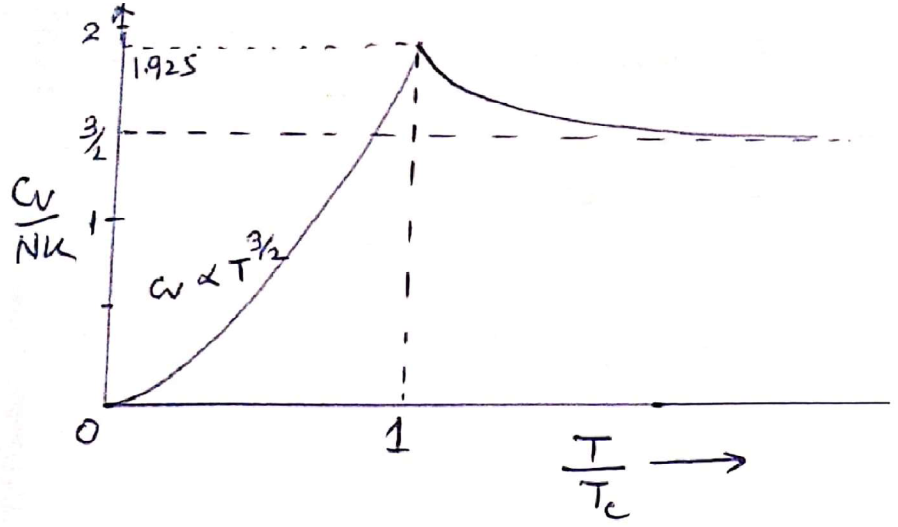
$$\boxed{\frac{C_v}{NK} = \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)}} \quad \text{when } T > T_c$$

In the classical limit $z \to 0$,

$$\therefore \frac{C_v}{NK} = \frac{15}{4} - \frac{9}{4} = \frac{3}{2} \text{ (ideal gas)}$$

For $z \to 1$ (when $T \to T_c$), $g_{1/2}(z)$ is divergent, so second term vanishes and

$$\frac{C_v}{NK} = \frac{15}{4}\cdot\frac{\zeta(5/2)}{\zeta(3/2)} = 1.925$$



when T/T_c is smaller than 1, C_V increases with temperature like $T^{3/2}$ to a maximum value. At $T = T_c$ a spike appears and when $T \to \infty$ C_V approaches $\frac{3}{2} NK$, value for an ideal gas. The apperence of spike is a signature for second order phase transition. (no latent heat). A Kink appears in the first derivative of E and a discontinuity appears in the second derivative of E i.e

$$\frac{\partial C_V}{\partial T} = \frac{\partial^2 E}{\partial T^2}.$$

References:

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- Elementary Statistical Physics by C. Kittel
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Thank You

For any questions/doubts/suggestions and submission of assignments

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